Handout 5: Going Partial II: Type Theory (Muskens 1995, Ch.6)
Semantics C Spring 2010

1. Applying partiality to our functional logic TY_2

TY_2^4 • a four-valued variant of TY_2

First simple tweak: let the domain of t include four values instead of two

$$D_t = \{T, F, N, B\}$$

Definition 1 (TY_2^4 frames)
A (standard) TY_2^4 frame is a set of sets $\{D_\alpha \mid \alpha \text{ is a functional type}\}$ such that...

$D_e \neq \varnothing$,
$D_s \neq \varnothing$,
$\mathbf{D_t = \{T, F, N, B\}}$ and
$D_{\alpha\beta}$ is the set of (total) functions from D_α to D_β.

Next, # and ★ are stipulated to be TY_2^4 formulae (type *t* terms). We use Strong Kleene operations (from LK4)[1] $-$, $\cap$, $\bigcap$, and $\bigcup$ to help with evaluation. And $\sqsubseteq$ is the ordering relation on L4.

Definition 2 (Tarski truth definition for TY_2^4)
The value $|A|^{M,a}$ of a term A on a TY_2^4 standard model $M = \langle\{D_\alpha\}_\alpha, I\rangle$ under an assignment *a* is defined as follows:

i. $|c| = I(c)$ if c is a constant;
$|x| = a(x)$ if x is a variable;

ii. $|\neg\varphi| = -|\varphi|$
(negation);

$|\varphi \wedge \psi| = |\varphi| \cap |\psi| = \min(\{|\varphi|, |\psi|\})$
(conjunction);

$|\#| = B$;
$|★| = N$;
(for Completeness)

iii. $|\forall x_\alpha\, \varphi|^{M,a} = \bigcap_{d\in D\alpha} |\varphi|^{M,a[d/x]}$
$= \min(\{|\varphi|^{M,a[d/x]}: d \in D_\alpha\})$
(universal quantification);

iv. $|A_{\alpha\beta}B_\alpha| = |A|(|B|)$
(function application);

v. $|\lambda x_\alpha\, A_\beta|^{M,a}$ = the $F \in D_{\alpha\beta}$ such that for all $d \in D_\alpha$: $F(d) = |A|^{M,a[d/x]}$
(lambda abstraction);

vi. $|A = B| = T$ if $|A| = |B|$
$= F$ if $|A| \neq |B|$
(identity).

Definition 3 (Entailment in TY_2^4)
Let Γ and Δ be sets of TY_2^4 formulae. The relation $\Gamma \models_s \Delta$ holds in TY_2^4 if

$\bigcap_{\varphi\in\Gamma} |\varphi|^{M,a} \sqsubseteq \bigcup_{\psi\in\Delta} |\psi|^{M,a}$ or, in other words... $\min(\{|\varphi|^{M,a}: \varphi \in \Gamma\}) \sqsubseteq \max(\{|\psi|^{M,a}: \psi \in \Delta\})$

for all TY_2^4 standard models *M* and assignments *a* to *M*.

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1 Refer to Extended Strong Kleene tables p.2 Handout4.

Question: Is this really a *partial* theory of types?

Answer: Yes- we have given up the classical connection between truth and falsity and are now using truth combinations to replace truth values.

Doubt: All of our functions in TY_2^4 are still *total* functions.
Worse, while some functions can be considered partial sets (ex: type *et*), others remain total objects no matter what way we look at them (ex: type *ee*).

2. Applying partiality to our relational logic TT_2

Recall the types of TT_2: The set of types is the smallest set of strings such that...

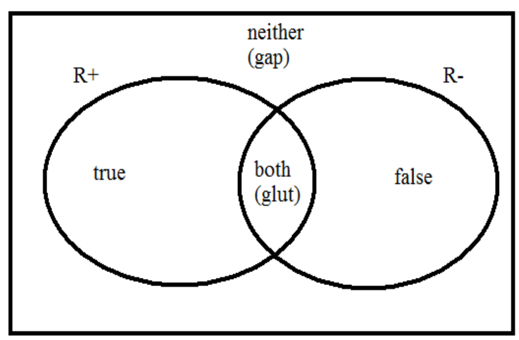
i. *e* (individuals) and *s* (world-time pairs) are types;
ii. if $\alpha_1,\ldots,\alpha_n$ are types ($n \geq 0$), then $<\alpha_1\ldots\alpha_n>$ is a type.

Since *t* is not a basic type in TT_2, we cannot simply replace the domain of truth values with the set of truth combinations and leave everything else as it was before, like we did to create TY_2^4.

Instead, we will partialize the objects that all non-basic domains consist of: relations! Below is the definition of a partial relation along with other relevant vocabulary.

Definition 4 (Partial relations)[2]
Let $D_1, \ldots, D_n$ be sets.

- An ***n-ary partial relation*** R on $D_1, \ldots, D_n$ is a tuple of relations $<R^+, R^->$ such that $R+, R- \subseteq D_1 \times \ldots \times D_n$.
- ***denotation***: the relation R^+ is called R's *denotation*;
- ***antidenotation***: the relation R^- is called R's *antidenotation*;
- ***gap***: the relation $(D_1 \times \ldots \times D_n) - (R^+ \cup R^-)$ written as $(R^+ \cup R^-)^c$)
- ***glut***: the relation $R^+ \cap R^-$



A partial relation is...

- ***coherent*** if its glut is empty.
- ***total*** if its gap is empty,
- ***incoherent*** if it is not coherent
- ***classical*** if it is both coherent and total.

- A unary partial relation is called a ***partial set***.
- If D is some set then the ***partial power set*** of D, PPow(D), is Pow(D)×Pow(D), that is, the set of all partial sets over D:
$$\{<R^+, R^-> \mid R^+, R^- \subseteq D\}.$$

Notice, in the relational theory a partialization of the relations in all non-basic domains leads to the desired shape of $D_{<>}$.

2 Note that the set notation used in Definition 4 corresponds to *actual* set notation– *not* the similar-looking Strong Kleene operations from LK4, which we used in Definitions 1-2.

Main Ideas:
For a tuple of objects and a partial relation R,

- it is true that they stand in R if they are in R's denotation;
- it is false that they stand in R if they are in R's antidenotation.

This leaves open the possibility that

- it is neither true nor false that a given tuple stand in R (they are in the gap)

  or
- that it is both true and false that they do (they are in the glut).

We now extend the Strong Kleene operations, used on the set of truth combinations {**T, F, N, B**}, to the class of partial relations.

Definition 5 (Operations on partial relations)
Let $\mathbf{R_1 = \langle R_1^+, R_1^- \rangle}$ and $\mathbf{R_2 = \langle R_2^+, R_2^- \rangle}$ be partial relations.
Define:

truth conditions | false conditions

$-\mathbf{R_1} := \langle R_1^-, R_1^+ \rangle$ **(partial complementation)**

$\mathbf{R_1 \cap R_2} := \langle R_1^+ \cap R_2^+, R_1^- \cup R_2^- \rangle$ **(partial intersection)**

$\mathbf{R_1 \cup R_2} := \langle R_1^+ \cup R_2^+, R_1^- \cap R_2^- \rangle$ **(partial union)**

$\mathbf{R_1 \subseteq R_2}$ iff $R_1^+ \subseteq R_2^+$ and $R_2^- \subseteq R_1^-$ **(partial inclusion)**

- For $\subseteq$, think material implication.

Let A be some set of partial relations.
Define:

truth conditions | false conditions

$\bigcap A := \langle \bigcap \{R^+ \mid R \in A\}, \bigcup \{R^- \mid R \in A\} \rangle$

$\bigcup A := \langle \bigcup \{R^+ \mid R \in A\}, \bigcap \{R^- \mid R \in A\} \rangle$

(Basically, generalized conjunction and disjunction over higher order types.)

Now that we've defined our tools, we can begin constructing a TT_2^4 logic.

Definition 6 (Frames) A *frame* is a set $\{D_\alpha \mid \alpha \text{ is a type}\}$ such that

$D_e \neq \varnothing$,

$D_s \neq \varnothing$ and

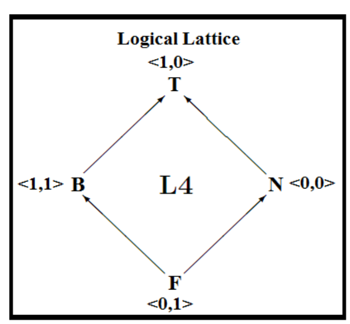
$D_{\langle \alpha 1 \ldots \alpha n \rangle} \subseteq PPow(D_{\alpha 1} \times \ldots \times D_{\alpha n})$.

A frame is *standard* if $D_{\langle \alpha 1 \ldots \alpha n \rangle} = PPow(D_{\alpha 1} \times \ldots \times D_{\alpha n})$ for all $\alpha_1, \ldots, \alpha_n$.

- That is, each domain $D_{\langle \alpha 1 \ldots \alpha n \rangle}$ consists of all the partial relations on domains $D_{\alpha 1}, \ldots, D_{\alpha n}$.

(We will only be working with standard frames.)

To get the set of truth-combinations {**T, F, N, B**} we need only to check the set PPow({∅}).



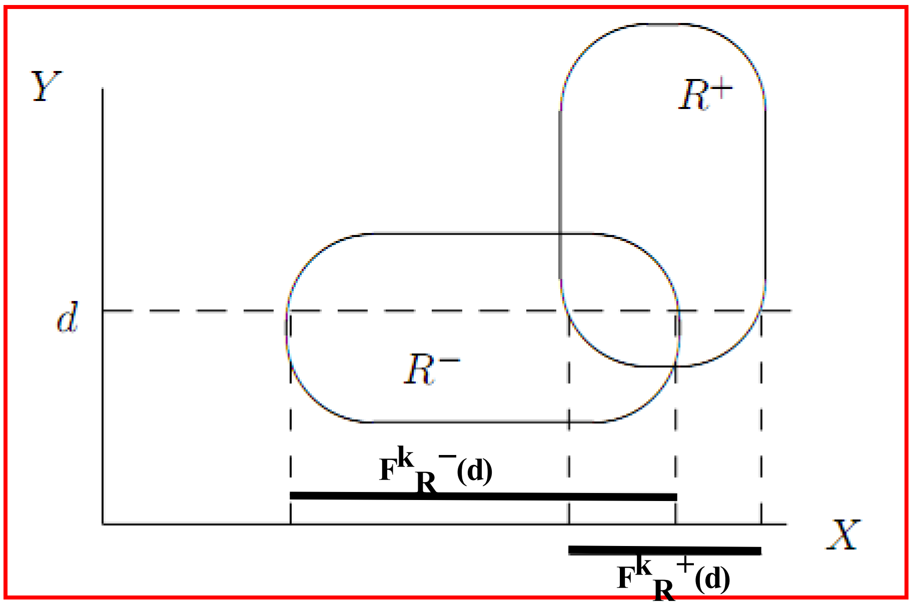
$$\mathbf{PPow(\{\varnothing\})} = Pow(\{\varnothing\}) \times Pow(\{\varnothing\})$$
$$= \{\langle\{\varnothing\}, \varnothing\rangle, \langle\varnothing, \{\varnothing\}\rangle, \langle\varnothing, \varnothing\rangle, \langle\{\varnothing\}, \{\varnothing\}\rangle\}$$
$$= \{\langle \mathbf{1, 0} \rangle, \langle \mathbf{0, 1} \rangle, \langle \mathbf{0, 0} \rangle, \langle \mathbf{1, 1} \rangle\}$$
$$= \{\ \mathbf{T},\ \mathbf{F},\ \mathbf{N},\ \mathbf{B}\ \}$$

- ➢If a value's first element is 1, it ***includes truth***.
- ➢If its second element is 1, it ***includes falsity***.

In TT_2, we decided to use only relational types rather than functional ones. To do this we relied on the existence of slice functions, which allow us to view any relation as a function. Now that we are going partial, we need to redefine our slice functions appropriately.

<u>Definition 7</u> (Slice Functions)
Let R be an n-ary partial relation and let $0 < k \leq n$.
The k-th slice function of R, $\boldsymbol{F^k_R}$, is defined by $\boldsymbol{F^k_R(d) = \langle F^k_R{}^+(d), F^k_R{}^-(d)\rangle}$.

Below is an example of the slice function of a binary partial relation.



A binary partial relation on the reals is now identified as a <u>pair of sets</u> (a partial set) in the Euclidean plane. This pair can be seen as a (total) function: for any point in the Y-axis, it returns a pair of sets of points on the X-axis. It is a function from points on the Y-axis to partial sets of points on the X-axis.

2.1 The logic of TT24

Finally, we are ready to give a Tarski definition evaluating the syntax of TT_2 on partial frames. The first order of business is to define our model.

A *very general model* is a tuple $\langle F, I\rangle$ where
- ➢$F = \{D_\alpha\}_\alpha$ is a partial frame and
- ➢I is an interpretation function for F.

We assume a *standard* frame (see Definition 6), and hence a *standard* very general model.

Next, we add logical constants # and ★ as type < > formulae. Again, we use Strong Kleene operations (from LK4) −, ∩, ⋂, and ∪ to help with evaluation. And ⊆ is the ordering relation on L4.

Definition 8 (Tarski truth definitions)
The value $||A||^{M,a}$ of a term A on a very general model *M* under an assignment *a* is defined as follows:

i. $||c|| = I(c)$ if c is a constant;
$||x|| = a(x)$ if x is a variable;

ii. $||\neg\varphi|| = -||\varphi||$
(negation);

$||\varphi \wedge \psi|| = ||\varphi|| \cap ||\psi||$
$= \langle ||\varphi^{+}|| \cap ||\psi^{+}||,\ ||\varphi^{-}|| \cup ||\psi^{-}|| \rangle$
(conjunction);

$||\#|| = \langle 1,1 \rangle$;
$||\bigstar|| = \langle 0,0 \rangle$;
(for Completeness)

iii. $||\forall x_\alpha\ \varphi||^{M,a} = \bigcap_{d \in D\alpha} ||\varphi||^{M,a[d/x]}$
$= \langle \bigcap_{d \in D\alpha} ||\varphi^{+}||^{M,a[d/x]},\ \bigcap_{d \in D\alpha} ||\varphi^{-}||^{M,a[d/x]} \rangle$
(universal quantification);

iv. $||A_{\alpha\beta}B_\alpha|| = F^1_{||A||}(||B||)$
(function application);

v. $||\lambda x_\alpha\ A_\beta||^{M,a}$ = the R such that
$F^1_R(d) = ||A||^{M,a[d/x]}$ for all $d \in D_\alpha$
(lambda abstraction);

vi. $||A = B|| = \langle 1,0 \rangle$ if $||A|| = ||B||$
$= \langle 0,1 \rangle$ if $||A|| \neq ||B||$
(identity).

(Recall from our original version of TT_2, that F^1 is simply the first slice function of some relation.)

Definition 9 (Entailment in TT_2^4)
Let Γ and Δ be sets of terms of some type $\alpha = \langle \alpha_1 \ldots \alpha_n \rangle$.
$\Gamma \models_s \Delta$, if

$$\bigcap_{A \in \Gamma} ||A||^{M,a} \subseteq \bigcup_{B \in \Delta} ||B||^{M,a}$$

or, in other words...

$$\langle \cap\{||A^{+}|| \mid A \in \Gamma\}, \cup\{||A^{-}|| \mid A \in \Gamma\} \rangle \subseteq \langle \cup\{||B^{+}|| \mid B \in \Delta\}, \cap\{||B^{-}|| \mid B \in \Delta\} \rangle$$

for all standard models *M* and assignments *a* to *M*.

2.2 Working through some examples

Non-basic Domains:
Ex.1:

Getting the domain <e> in TT_2
$D_{\langle e \rangle} = Pow(D_e)$
$= \{X \mid X \subseteq D_e\}$
$=$ GREEN$_{\langle e \rangle}$, GHOST$_{\langle e \rangle}$, IDEA$_{\langle e \rangle}$, SHOE$_{\langle e \rangle}$ ...

Getting the domain <e> in TT_2^4 (c.f. Definition 6)
$D_{\langle e \rangle} = PPow(D_e)$
$= \{\langle X,Y \rangle \mid X,Y \subseteq D_e\}$
$= \langle$ GREEN$^{+}_{\langle e \rangle}$, GREEN$^{-}_{\langle e \rangle} \rangle$,
$\langle$ GHOST$^{+}_{\langle e \rangle}$, GHOST$^{-}_{\langle e \rangle} \rangle$,
$\langle$ IDEA$^{+}_{\langle e \rangle}$, IDEA$^{-}_{\langle e \rangle} \rangle$, ...

Slice Functions:
$||\lambda x_\alpha\ A_\beta||^{M,a}$ = the R such that $F^1_R(d) = ||A||^{M,a[d/x]}$ for all $d \in D_\alpha$ (c.f. Definition 8)

Ex.2: $||\lambda x_e \text{GREEN}(x)||^{M,a}$ **= the R such that** $F^1_R(d) = ||\text{GREEN}||^{M,a[d/x]}$ **for all** $d \in D_e$

in TT_2
$F^1_{||\text{GREEN}||}(d)$

in TT_2^4
$F^1_{||\text{GREEN}||}(d) = \langle F^1_{||\text{GREEN}||+}(d), F^1_{||\text{GREEN}||-}(d) \rangle$

Function application:
$||A_{\alpha\beta}B_{\alpha}|| = F^1_{||A||}(||B||)$

Remember: Here, 0 is just ∅ and
1 is just {∅}.

Ex.2: "John is green."

$\text{GREEN}_{<e>}(\text{JOHN}_e) = F^1_{||\text{GREEN}||}(||\text{JOHN}||) = F^1_{I(\text{GREEN})}(I(\text{JOHN})) = F^1_{\text{GREEN}}($🚹$)$

in TT_2

$F^1_{\text{GREEN}}($🚹$) = 0$

F^1_{GREEN}

🚹	→ 0
👽	→ 1
🗨	→ 0
	…

in TT_2^4

$F^1_{\text{GREEN}}($🚹$) = \langle F^1_{\text{GREEN}+}($🚹$), F^1_{\text{GREEN}-}($🚹$)\rangle$
$= \langle 0, 1\rangle$

$F^1_{\text{GREEN}+}$

🚹	→ 0
👽	→ 1
🗨	→ 0
	…

(denotation)

$F^1_{\text{GREEN}-}$

🚹	→ 1
👽	→ 0
🗨	→ 0
	…

(antidenotation)

Ex.3: "Abe is furious."

$\text{FURIOUS}_{<e>}(\text{ABE}_e) = F^1_{||\text{FURIOUS}||}(||\text{ABE}||) = F^1_{I(\text{FURIOUS})}(I(\text{ABE})) = F^1_{\text{FURIOUS}}($👽$)$

in TT_2

$F^1_{\text{FURIOUS}}($👽$) = 1$

F^1_{FURIOUS}

🚹	→ 1
👽	→ 1
🗨	→ 0
	…

in TT_2^4

$F^1_{\text{FURIOUS}}($👽$) = \langle F^1_{\text{FURIOUS}+}($👽$), F^1_{\text{FURIOUS}-}($👽$)\rangle$
$= \langle 1, 1\rangle$

$F^1_{\text{FURIOUS}+}$

🚹	→ 1
👽	→ 1
🗨	→ 0
	…

(denotation)

$F^1_{\text{FURIOUS}-}$

🚹	→ 0
👽	→ 1
🗨	→ 0
	…

(antidenotation)

Conjunction:
$||\varphi \wedge \psi|| = ||\varphi|| \cap ||\psi||$

Ex.4: "John is green and Abe is furious."

$\text{GREEN}(\text{JOHN}_e) \wedge \text{FURIOUS}(\text{ABE}_e)$

in TT_2

$||\text{GREEN}(\text{JOHN}_e) \wedge \text{FURIOUS}(\text{ABE}_E)||$
$= ||\text{GREEN}(\text{JOHN}_e)|| \cap ||\text{FURIOUS}(\text{ABE}_e)||$
$= \varnothing \cap \{\varnothing\}$
$= \varnothing$
$= 0$

in TT_2^4

$||\text{GREEN}(\text{JOHN}_e) \wedge \text{FURIOUS}(\text{ABE}_E)||$
$= ||\text{GREEN}(\text{JOHN}_e)|| \cap ||\text{FURIOUS}(\text{ABE}_e)||$
$= \langle F^1_{\text{GREEN}+}($🚹$) \cap F^1_{\text{FURIOUS}+}($👽$), F^1_{\text{GREEN}-}($🚹$) \cup F^1_{\text{FURIOUS}-}($👽$)\rangle$
$= \langle \varnothing \cap \{\varnothing\}, \{\varnothing\} \cup \{\varnothing\}\rangle$
$= \langle \varnothing, \{\varnothing\}\rangle$
$= \langle 0, 1\rangle$