# Handout 4: Going Partial (Muskens 1995, Ch. 5)

Semantics C (Spring 2010)

One of the basic assumptions of classical logic is the assumption that a sentence is false if and only if it is not true. Thus two possibilities are excluded:

(a) that sentences are neither true nor false;
(b) that sentences are both true and false.

Partial logics are logics in which at least one of these possibilities is allowed. This handout has two goals:

- develop partial variants of classical propositional logic and classical predicate logic.
- Generalize these to a full partial theory of types.

**The First Consideration:**
Should we allow both (a) and (b) or will it do to allow just (a) or just (b)?

- Between the last two possibilities there is not much to choose, since under certain reasonable assumptions a logic that allows sentences to be both true and false but does not allow them to be neither will be isomorphic to a logic that does the reverse.
- But the choice between a logic that allows all four possible combinations of truth values and one that allows only three of them is real.

Muskens allows both overdefinedness and underdefinedness (Four-valued logic).

- Accepting one, but not the other, of the two symmetrical possibilities (a) and (b) above would introduce certain asymmetries into the logic that are less than nice. Four-valued logics tend to be more elegant than three-valued ones.
- If we want to give a natural account of the semantics of the psychological verbs, we need situations that are overdefined as well as situations that are underdefined.
  - Good examples of underdefined situations are people's visual scenes, the situations they see at some given moment.
  - But people do not only see things, they imagine and believe things as well. People can believe things to be and not to be the case, or at least they can believe that they believe this.[1] It is therefore useful to have overdefined situations around if you want to model the logic of the verbs believe, imagine, dream and the like.

Now that the tie between truth and falsity has been cut, we are left with four combinations of truth values:[2]

- 'true and not false' **T**

[1]The sentences we usually encounter in Montague Grammar, sentences not containing truth predicates etc., are always either true or false and never both in the real world (but they may be neither true nor false in parts of the real world or both true and false in imagined situations). In the next chapter we shall make sure that this is the case in all models under consideration by means of a meaning postulate. Of course the situation may be different with respect to the Liar sentence, which is treated as neither true nor false or as both true and false in most contemporary theories. Although the logic that is developed below is a higher order generalization of the four-valued predicate logic that is usually employed in analyses of the Liar paradox, we postpone a treatment of the Liar within the present framework to another occasion.

[2]Note that **T**, **F**, **N** and **B** are not truth values themselves, they are *truth combinations*.

- ‘false and not true’ **F**
- ‘neither true nor false’ **N**
- ‘both true and false’ **B**

Say that:

- a combination **X** *includes truth* iff **X** equals **T** or **B**
- a combination **X** *includes falsity* iff **X** equals **F** or **B**.[3]

## Partial Propositional Logic

How does the truth combination of a complex sentence depend on the truth combinations of its parts? There is a stunningly simple answer to this question.

- Truth and falsity can be computed just as it is done ordinarily, provided that truth conditions and falsity conditions are separated:

  i. $\neg\varphi$ is true if and only if $\varphi$ is false,
  $\neg\varphi$ is false if and only if $\varphi$ is true;

  ii. $\varphi \wedge \psi$ is true if and only if $\varphi$ is true and $\psi$ is true,
  $\varphi \wedge \psi$ is false if and only if $\varphi$ is false or $\psi$ is false;

  iii. $\varphi \vee \psi$ is true if and only if $\varphi$ is true or $\psi$ is true,
  $\varphi \vee \psi$ is false if and only if $\varphi$ is false and $\psi$ is false.

**An Example:**
If $\varphi$ receives the combination **N** and $\psi$ gets the combination **T**, then $\varphi \wedge \psi$ gets **N**: $\varphi \wedge \psi$ is not true since $\varphi$ isn't and it is not false since neither $\varphi$ nor $\psi$ is. THUS:

| $\wedge$ | **T** | **F** | **N** | **B** |
|---|---|---|---|---|
| **T** | **T** | **F** | **N** | **B** |
| **F** | **F** | **F** | **F** | **F** |
| **N** | **N** | **F** | **N** | **F** |
| **B** | **B** | **F** | **F** | **B** |

| $\vee$ | **T** | **F** | **N** | **B** |
|---|---|---|---|---|
| **T** | **T** | **T** | **T** | **T** |
| **F** | **T** | **F** | **N** | **B** |
| **N** | **T** | **N** | **N** | **T** |
| **B** | **T** | **B** | **T** | **B** |

| $\neg$ | |
|---|---|
| **T** | **F** |
| **F** | **T** |
| **N** | **N** |
| **B** | **B** |

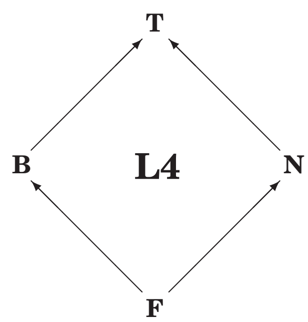
Note that if we restrict these tables to {**T**,**F**,**N**} we obtain just the Strong Kleene tables; following  we shall therefore call the above the *Extended Strong Kleene* tables.

- The two-place function on {**T**,**F**,**N**,**B**} given by the table for conjunction will be denoted by $\cap$ and similarly $\cup$ will denote the function given by the table for disjunction.
- the one-place function given by the negation table we shall denote with $-$. We write $\mathbf{X} \subseteq \mathbf{Y}$ iff for some **Z**, $\mathbf{X} \cup \mathbf{Z}$ equals **Y**.
  - Notice that $\mathbf{X} \subseteq \mathbf{Y}$ if and only if **Y** includes truth if **X** does and **X** includes falsity if **Y** includes falsity.

[3] **T**, **F**, **N** and **B** are often defined as $\{true\}$, $\{false\}$, $\emptyset$ and $\{true, false\}$ respectively. A truth combination **X** can then be said to include truth (falsity) if $true \in \mathbf{X}$ ($false \in \mathbf{X}$). This is elegant but I shall not adopt this definition for the reason that union and intersection on these sets do not correspond to disjunction and conjunction respectively. (They do however correspond to the operations $\sqcup$ and $\sqcap$ to be defined below.)

- The partial order $\subseteq$, a generalization of the ordering relation in the Boolean algebra $\{true, false\}$, will play the role of *entailment* in the logics to be defined below

The operations $\cap$ and $\cup$ form a distributive lattice on the set of truth combinations, depicted below as L4. Adding negation to L4 gives a structure that conforms to almost all of the customary axioms of the theory of Boolean algebra's.[4]




Logical lattice

To be more precise, the structure LK4 = $\langle\{\mathbf{T},\mathbf{F},\mathbf{N},\mathbf{B}\}, \cap, \cup, -, \mathbf{T}, \mathbf{F}\rangle$ satisfies the following list of axioms:

**The Axioms for Distributive Lattices:** (also called a *Kleene algebra*)

i. $0 + a = a$, $1 \cdot a = a$;

ii. $a'' = a$;

iii. $(a \cdot b)' = a' + b'$, $(a + b)' = a' \cdot b'$,

iv. $0' = 1$, $1' = 0$.

There are two Boolean axioms that do not hold in LK4 and that, of course, we don't want to hold:

- $a + a' = 1$
- $a \cdot a' = 0$.

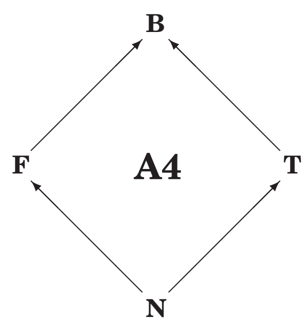
If we would add them to the above list of axioms we would get a full set of axioms for Boolean algebras; the move from single truth values to combinations of them did not cause an unmotivated loss of algebraical properties.

**A Second Algebra**

Let us say that a truth combination $\mathbf{X}$ *approximates* a truth combination $\mathbf{Y}$, $\mathbf{X} \sqsubseteq \mathbf{Y}$, if and only if $\mathbf{Y}$ includes truth if $\mathbf{X}$ does and $\mathbf{Y}$ includes falsity if $\mathbf{X}$ includes falsity.

- Intuitively, if $\mathbf{X} \sqsubseteq \mathbf{Y}$ then $\mathbf{Y}$ contains at least as much information as $\mathbf{X}$ does.
- The relation $\sqsubseteq$ on the truth combinations gives rise to the distributive lattice A4 below.

[4]Note that this operation corresponds with rotating L4 halfway around its **B**-**N** axis




Approximation lattice

Clearly, A4 is just L4 put on its side.

- Its meet and join correspond to two natural ways to combine information.
- If one good friend tells you that a certain rumour is true and another equally good friend tells you that it is false, there are two evenhanded ways to combine the information that they have given you.
  - Accept that the rumour is both true and false.
  - Conclude that the rumour is neither.

These join and meet operations are what we can call *half-duals* of the operations $\cup$ and $\cap$ defined above.

- The meet of two truth combinations **X** and **Y** in A4, written $\mathbf{X} \sqcap \mathbf{Y}$, includes truth (falsity) iff both **X** and **Y** include truth (falsity).
- The join $\mathbf{X} \sqcup \mathbf{Y}$, includes truth (falsity) iff either **X** or **Y** includes truth (falsity).

We can also define a half-dual of negation:

- This is the unary operation $\lrcorner$ that leaves **T** and **F** fixed, but swaps **B** and **N**. That is, $\lrcorner\mathbf{X}$ includes truth (falsity) iff **X** does not include falsity (truth).

The structure $\langle\{\mathbf{T},\mathbf{F},\mathbf{N},\mathbf{B}\}, \sqcap, \sqcup, \lrcorner, \mathbf{B}, \mathbf{N}\rangle$, which we shall call AK4, is a Kleene algebra.

At this point we may introduce propositional connectives corresponding to our new operations on truth combinations. Let us write:

- $\otimes$ for $\sqcap$
- $\oplus$ for $\sqcup$
- $\lrcorner$ for $\lrcorner$

| $\otimes$ | **T** | **F** | **N** | **B** |
|---|---|---|---|---|
| **T** | **T** | **N** | **N** | **T** |
| **F** | **N** | **F** | **N** | **F** |
| **N** | **N** | **N** | **N** | **N** |
| **B** | **T** | **F** | **N** | **B** |

| $\oplus$ | **T** | **F** | **N** | **B** |
|---|---|---|---|---|
| **T** | **T** | **B** | **T** | **B** |
| **F** | **B** | **F** | **F** | **B** |
| **N** | **T** | **F** | **N** | **B** |
| **B** | **B** | **B** | **B** | **B** |

| $\lrcorner$ | |
|---|---|
| **T** | **T** |
| **F** | **F** |
| **N** | **B** |
| **B** | **N** |

Back up and look at our new connectives.

- The connective $\otimes$ is what  calls *interjunction*. It has the truth conditions of $\wedge$ but the falsity conditions of $\vee$.
- Its dual $\oplus$ has the truth conditions of $\vee$ and the falsity conditions of $\wedge$.
- Entailment corresponds to the L4 ordering while approximation is the A4 ordering, we can even say that approximation is the half-dual of entailment.

We see here that our decision to severe the classical connection between truth and falsity does not only lead to a strengthening of the entailment relation, it also leads to a complete set of half-duals of our ordinary logical concepts.

**Towards a Partial Propositonal Logic:**
In order to define a partial propositional logic we need to augment the syntax of ordinary two-valued propositional logic with new connectives. Apart from the classical $\wedge$ and $\neg$, these will be:

- # is a constant (of formula type) to denote **B**
- $\star$ is a constant (of formula type) to denote **N**
- = is a two-place identity connective[5]

The choice of = as a primitive symbol rather than $\otimes$ and $\lrcorner$ is taken with a view to the generalization to the theory of types that we are after: in type theory we want to be able to express that objects of equal type are identical whatever this type is, so we shall need a symbol to express identity between objects of formula type $\langle\rangle$ anyhow.

Having augmented the syntax of propositional logic thus, we can now give a partial semantics.

**Definition 1** *(Tarski truth definition for partial propositional logic) Let $L$ be a set of propositional constants and let $V : L \to \{\mathbf{T}, \mathbf{F}, \mathbf{N}, \mathbf{B}\}$ be a (*valuation*)* function. *The* value $|\varphi|^V$ *of a formula $\varphi$ under $V$ is defined as follows:*

*i.* $|p|^V = V(p)$ *if* $p \in L$;

*ii.* $|\neg\varphi|^V = -|\varphi|^V$;
$|\varphi \wedge \psi|^V = |\varphi|^V \cap |\psi|^V$;
$|\#|^V = \mathbf{B}$;
$|\star|^V = \mathbf{N}$;
$|\varphi = \psi|^V = \mathbf{T}$ *if* $|\varphi|^V = |\psi|^V$
$\qquad\qquad = \mathbf{F}$ *if* $|\varphi|^V \neq |\psi|^V$;

The clauses for negation, conjunction, # and $\star$ are clearly in agreement with the preceding discussion; the clause for = we shall discuss shortly. We can now define $\otimes$, $\oplus$ and $\lrcorner$ and the classical abbreviatory symbols from the primitive connectives that we have chosen.

**Definition 2** *(Abbreviations) Write*

| | | |
|---|---|---|
| $\top$ | *for*[6] | $\star = \star$ |
| $\bot$ | *for* | $\neg\top$ |
| $\varphi \vee \psi$ | *for* | $\neg(\neg\varphi \wedge \neg\psi)$ |
| $\varphi \to \psi$ | *for* | $\neg\varphi \vee \psi$ |
| $\varphi \leftrightarrow \psi$ | *for* | $(\varphi \to \psi) \wedge (\psi \to \varphi)$ |
| $\varphi\,@\,\psi$ | *for* | $(\varphi \wedge \#) \vee (\psi \wedge \star)$ |
| $\varphi \otimes \psi$ | *for* | $(\varphi \wedge \psi)\,@\,(\varphi \vee \psi)$ |
| $\varphi \oplus \phi$ | *for* | $(\varphi \vee \psi)\,@\,(\varphi \wedge \psi)$ |
| $\lrcorner\varphi$ | *for* | $\neg(\varphi = \bot) \wedge$ $((\varphi = \star) \to \#) \wedge ((\varphi = \#) \to \star)$ |

[5] Notice that $\leftrightarrow$, if we give it its usual definition in terms of $\wedge$ and $\neg$, can no longer do duty as identity as it does in two-valued propositional logic, since $\varphi \leftrightarrow \psi$ gets the value **N** if both $\varphi$ and $\psi$ get the value **N**.

[6] Warning: while $\top$ is often used to denote the top element of the approximation lattice A4, we follow the established practice to let it

Notice that $\varphi\,@\,\psi$ has the truth conditions of $\varphi$ but the falsity conditions of $\psi$ and hence the right operations are assigned to $\otimes$ and $\oplus$.

**Entailment:**
Let us say that a valuation $V$ *verifies* (*falsifies*) a formula $\varphi$ if $|\varphi|^V$ includes truth (falsity). Letting $\bigcap$ and $\bigcup$ denote the infimum and the supremum of sets of truth combinations in the logical lattice L4, we define entailment thus:

**Definition 3** *(Entailment in partial propositional logic) Let* $\Gamma$ *and* $\Delta$ *be sets of formulae.* $\Gamma$ entails $\Delta$*,* $\Gamma \models \Delta$*, if*

$$\bigcap_{\varphi\in\Gamma} |\varphi|^V \subseteq \bigcup_{\psi\in\Delta} |\psi|^V$$

*for all valuations* $V$*.*

The resulting notion of logical consequence is double-barrelled: A set of premises entails a set of conclusions if and only if two conditions hold.

- The first of these is that each valuation that verifies all premises verifies some conclusion.
- the second is that each valuation that falsifies all conclusions falsifies some premise.
- Thus, logical consequence is preservation of truth in one direction and preservation of falsity in the other.

**Other stabs at Entailment:**
There are two obvious single-barrelled alternatives to the one above.

- Write $\varphi \models' \psi$ iff each valuation that verifies $\varphi$ verifies $\psi$.
- Write $\varphi \models'' \psi$ iff each valuation that falsifies $\psi$ falsifies $\varphi$.

We do not use these because there are some important properties that $\models$ does, but that $\models'$ and $\models''$ do not, share with the classical notion of entailment.

- Contraposition: $\varphi \models \psi$ if and only if $\neg\psi \models \neg\varphi$.
  - On the other hand neither $\models'$ nor $\models''$ satisfies this property.
  - $\star \models' \bot$ but not that $\neg\bot \models' \neg\star$.
- Classical Logical Equivalence: $\varphi \cong \psi$ if and only if $\varphi \models \psi$ and $\psi \models \varphi$
  since $|\varphi|^V = |\psi|^V$ for each valuation $V$.
  - The other notions fail since both $\star \models' \bot$ and $\bot \models' \star$, but of course $\star$ and $\bot$ get different values.
- Classical Correspondence: $\varphi \models \psi$ if and only if $\varphi \cong \varphi \wedge \psi$ if and only if $\psi \cong \varphi \vee \psi$.
  - Again this fails for the other relations; we have for example that $\star \models' \bot$ but not that $\star \cong \star \wedge \bot$.

We see here that an adoption of $\models'$ or of $\models''$ as our notion of logical consequence would lead to deviations from classical logic that are unaccounted for by our original motivation of cutting the tie between truth and falsity.

Also, we don't lose much since we can recover $\models'$ and $\models''$ in terms of $\models$:

$$\varphi \models' \psi \text{ if and only if } \varphi \models \psi, \star$$

$$\varphi \models'' \psi \text{ if and only if } \varphi, \star \models \psi.$$

**Identity:**
This clause in Definition 1 seems the only reasonable one, given the constraint that identity statements $\varphi = \psi$ are two-valued, that is, if they always get the value **T** or **F**.

be the formula that is always true and never false, that is, it denotes the top of the logical lattice L4; a similar remark can be made with respect to the symbol $\bot$.

- But what about this requirement? Aren't there reasonable definitions of the semantics of = such that an identity formula $\varphi = \psi$ could in some cases get a value **N** or **B**?
- The answer is no. The possibility is ruled out by the requirement that identity should satisfy the identity axioms, that is, the demand that the following should hold:

$$\models \varphi = \varphi$$

$$\varphi = \psi, [\psi/p]\chi \models [\varphi/p]\chi.$$

Suppose that in Definition 28 we had adopted some alternative clause for =, but that the two statements above would still hold. We derive that even then no formula $\varphi = \psi$ would get **B** or **N** as a value under any valuation.

- From $\models \varphi = \varphi$ it follows that $\varphi = \varphi$ must always get the value **T** and hence that $\neg(\varphi = \varphi)$ must get the value **F**.
- An instantiation of the other statement gives $\varphi = \psi$, $\neg(\varphi = \psi) \models \neg(\varphi = \varphi)$. From the assumption that $|\varphi = \psi|^V = \mathbf{N}$ for some $V$ we derive $\mathbf{N} \subseteq \mathbf{F}$, which is absurd.
- From the assumption that $|\varphi = \psi|^V = \mathbf{B}$ we derive that $\mathbf{B} \subseteq \mathbf{F}$, no less absurd.
- We see that if = is to satisfy the identity axioms it has to be two-valued.

**Functional Completeness:**
If $f$ is some $n$-ary truth function $f : \{\mathbf{T},\mathbf{F},\mathbf{N},\mathbf{B}\}^n \to \{\mathbf{T},\mathbf{F},\mathbf{N},\mathbf{B}\}$ and $\varphi$ is some formula in which exactly the propositional constants $p_1, \ldots, p_n$ occur, we say that $\varphi$ *expresses* $f$ iff $|\varphi|^V = f(V(p_1),\ldots,V(p_n))$ for each valuation $V$. The following proposition holds (see the Appendix for its proof).

**Theorem 1** *Every truth function is expressed by some formula.*

As a corollary we find that the set $\{\otimes, \lrcorner, \wedge, \neg\}$ is expressively adequate as well, since $\star$ is equivalent with $p_0 \otimes \neg \lrcorner p_0$, # is equivalent with $\lrcorner\star$ and

$$((\varphi \,@ \lrcorner\varphi) \leftrightarrow (\psi \,@ \lrcorner\psi)) \wedge ((\lrcorner\varphi \,@\, \varphi) \leftrightarrow (\lrcorner\psi \,@\, \psi))$$

is equivalent with $\varphi = \psi$. The result is that we can express all functions on the four truth combinations.

Other authors want only pretty truth functions and consider on the family of monotonic functions.

**Definition 4** *(Monotonic truth functions) An n-ary truth function $f$ is called* monotonic *if it holds that*

$$x_1 \sqsubseteq y_1, \ldots, x_n \sqsubseteq y_n \Rightarrow f(x_1,\ldots,x_n) \sqsubseteq f(y_1,\ldots,y_n)$$

*for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in \{\mathbf{T},\mathbf{F},\mathbf{N},\mathbf{B}\}$.*

There is a nice correspondence between monotonic truth functions and sentences that are built up using only $\star$, # and the classical $\wedge$, $\neg$ and $\top$.

**Theorem 2** *(Proved in the Appendix) A truth function is monotonic if and only if it is expressed by a sentence that is built up from propositional constants using only $\star$, #, $\wedge$, $\neg$ and $\top$.*

The monotonic truth functions are a natural enough class to consider, but we do not want to be restricted. We want not monontoinc functions, like identity, as well as the following:

**Definition 5** *(More abbreviations)*

| | | |
|---|---|---|
| $\varphi \twoheadrightarrow \psi$ | *abbreviates* | $(\varphi \wedge \psi) = \varphi$ |
| $\varphi \sqsubseteq \psi$ | *abbreviates* | $(\varphi \otimes \psi) = \varphi$ |

Clearly these are object-language versions of the ordering relations $\subseteq$ and $\sqsubseteq$ in the lattices L4 and A4 discussed above.

| $\twoheadrightarrow$ | **T** | **F** | **N** | **B** |
|---|---|---|---|---|
| **T** | **T** | **F** | **F** | **F** |
| **F** | **T** | **T** | **T** | **T** |
| **N** | **T** | **F** | **T** | **F** |
| **B** | **T** | **F** | **F** | **T** |

| $\sqsubseteq$ | **T** | **F** | **N** | **B** |
|---|---|---|---|---|
| **T** | **T** | **F** | **F** | **T** |
| **F** | **F** | **T** | **F** | **T** |
| **N** | **T** | **T** | **T** | **T** |
| **B** | **F** | **F** | **F** | **T** |

Since the double-headed arrow corresponds to the relation $\subseteq$ on the truth combinations it also corresponds to entailment; we have:

$$\varphi \models \psi \text{ if and only if } \models \varphi \twoheadrightarrow \psi.$$

The truth table for $\twoheadrightarrow$ given above is just the one that Anderson and Belnap associate with the notion of 'tautological entailment'.

- One of our motives to partialize classical logic has been our discontent with the coarse-grainedness of the classical relation of entailment.
- Some arguments that are classically valid are nevertheless irrelevant. (Examples are: $\models \varphi \vee \neg\varphi$; $\varphi \wedge \neg\varphi \models$ and $\varphi \wedge \neg\varphi \models \psi \vee \neg\psi$.)
- Since this irrelevance of classical entailment leads to problems if we use classical logic as an instrument to describe the semantics of natural language, we have wanted to get rid of it.
- The source of the irrelevance we have diagnosed to be the classical connection between truth and falsity. Giving up this connection indeed leads to a form of Relevant Logic.

## Partial Predicate Logic

Assuming some countably infinite set of individual variables and a set $L$ (a language) consisting of relation symbols (each having a fixed number of argument places) and individual constants (we omit function symbols):

- *terms* to are either individual constants or individual variables.
- The set of formulae we define as follows:

**Definition 6** *The set of (*partial predicate logical*)* formulae *is defined with the following clauses:*

*i.* *If $R$ is an $n$-ary relation symbol and $t_1, \ldots, t_n$ are terms then $Rt_1 \ldots t_n$ is an (atomic) formula;*

*ii.* *If $t_1$ and $t_2$ are terms then $(t_1 = t_2)$ is an (atomic) formula;*

*iii.* *$\#$ and $\star$ are (atomic) formulae;*

*iv.* *If $\varphi$ and $\psi$ are formulae then $\neg\varphi$, $(\varphi \wedge \psi)$ and $(\varphi = \psi)$ are formulae;*

*v.* *If $\varphi$ is a formula and $x$ is an individual variable, then $\forall x \varphi$ is a formula.*

Note that the identity symbol may occur in two different contexts; in one it is intended to denote identity between individuals, in the other it denotes identity between combinations of truth values.

**The Semantics:**
We define an *interpretation function* for a set $D$ in the standard way.

- It is a function $I$ with domain $L$ such that $I(c) \in D$ if $c$ is an individual constant and $I(R) \in D^n$ if $R$ is an $n$-ary relation symbol.

- But we need *two* interpretation functions for each model now: a *model* for partial predicate logic is a triple $\langle D, I^+, I^- \rangle$ where $I^+$ and $I^-$ are interpretation functions for $D$.
- We stipulate that $I^+(c) = I^-(c)$ if $c$ is an individual constant, but for relational symbols $R$ the *denotation* $I^+(R)$ and the *antidenotation* $I^-(R)$ may differ:
  - The denotation of a relation symbol consists of those tuples for which it is *true* that they stand in the relation.
  - The antidenotation consists of the tuples for which this is *false*.
- As before truth and falsity are not connected in the classical way, it may be neither true nor false or it may be both true and false that some tuple stands in a certain relation.

*Assignments* for a given model $\langle D, I^+, I^- \rangle$ are defined as usual:

- they are functions taking variables to elements of $D$. If $a$ is an assignment then $a[d/x]$ is the assignment such that $a[d/x](x) = d$ and $a[d/x](y) = a(y)$ if $y \neq x$.
- The *value* of a term $t$ in a model $M = \langle D, I^+, I^- \rangle$ under an assignment $a$, $|t|^{M,a}$, is defined as $I^+(t)$ (or, equivalently, as $I^-(t)$) if $t$ is a constant and as $a(t)$ if $t$ is a variable.

We can now define truth.

**Definition 7** *(Tarski truth definition for partial predicate logic) The* value $|\varphi|^{M,a} \in \{\mathbf{T}, \mathbf{F}, \mathbf{N}, \mathbf{B}\}$ *of a formula $\varphi$ on a model $M = \langle D, I^+, I^- \rangle$ under an assignment $a$ is defined in the following way (I suppress superscripts where possible):*

*i.* $|Rt_1 \dots t_n| = \mathbf{T}$ *if* $\langle |t_1|, \dots, |t_n| \rangle \in I^+(R) - I^-(R)$,
$|Rt_1 \dots t_n| = \mathbf{F}$ *if* $\langle |t_1|, \dots, |t_n| \rangle \in I^-(R) - I^+(R)$,
$|Rt_1 \dots t_n| = \mathbf{N}$ *if* $\langle |t_1|, \dots, |t_n| \rangle \in D^n - (I^+(R) \cup I^-(R))$,
$|Rt_1 \dots t_n| = \mathbf{B}$ *if* $\langle |t_1|, \dots, |t_n| \rangle \in I^+(R) \cap I^-(R)$;

*ii.* $|t_1 = t_2| = \mathbf{T}$ *if* $|t_1| = |t_2|$,
$|t_1 = t_2| = \mathbf{F}$ *if* $|t_1| \neq |t_2|$;

*iii.* $|\#| = \mathbf{B}$;
$|\star| = \mathbf{N}$;

*iv.* $|\neg\varphi| = -|\varphi|$;
$|\varphi \wedge \psi| = |\varphi| \cap |\psi|$;
$|\varphi = \psi| = \mathbf{T}$ *if* $|\varphi| = |\psi|$,
$|\varphi = \psi| = \mathbf{F}$ *if* $|\varphi| \neq |\psi|$;

*v.* $|\forall x \varphi|^{M,a} = \bigcap_{d \in D} |\varphi|^{M,a[d/x]}$.

There is also the following equivalent formulation (We write $M \models \varphi[a]$ if $|\varphi|^{M,a}$ includes truth and $M \Dashv \varphi[a]$ if $|\varphi|^{M,a}$ includes falsity).

i. $M \models Rt_1 \dots t_n[a]$ iff $\langle |t_1|, \dots, |t_n| \rangle \in I^+(R)$;
$M \Dashv Rt_1 \dots t_n[a]$ iff $\langle |t_1|, \dots, |t_n| \rangle \in I^-(R)$;

ii. $M \models t_1 = t_2[a]$ iff $|t_1| = |t_2|$
$M \Dashv t_1 = t_2[a]$ iff $|t_1| \neq |t_2|$;

iii. $M \models \#[a]$
$M \Dashv \#[a]$;
$M \not\models \star[a]$
$M \not\Dashv \star[a]$;

iv. $M \models \neg\varphi[a]$ iff $M =\!\!| \; \varphi[a]$;
$M =\!\!| \; \neg\varphi[a]$ iff $M \models \varphi[a]$
$M \models \varphi \wedge \psi[a]$ iff $M \models \varphi[a]$ and $M \models \psi[a]$
$M =\!\!| \; \varphi \wedge \psi[a]$ iff $M =\!\!| \; \varphi[a]$ or $M =\!\!| \; \psi[a]$;
$M \models \varphi = \psi[a]$ iff $|\varphi| = |\psi|$
$M =\!\!| \; \varphi = \psi[a]$ iff $|\varphi| \neq |\psi|$;

v. $M \models \forall x \varphi[a]$ iff $M \models \varphi[a[d/x]]$ for all $d \in D$
$M =\!\!| \; \forall x \varphi[a]$ iff $M =\!\!| \; \varphi[a[d/x]]$ for some $d \in D$.

Note that the definition does not only give the Extended Strong Kleene truth conditions to $\neg$ and $\wedge$, but to $\forall$ now as well:

- A universally quantified formula is true iff all its instances are true, false iff at least one of its instances is false. Again this condition is in fact the classical one; of course in the classical case one of the condition's conjuncts is superfluous.
- It is clear that clauses iii. and iv. are the only ones possible if we want the logic to be a generalization of the partial propositional logic discussed above.
- The form of clause i. is a direct consequence of the denotation-antidenotation approach.
- Clause ii. (a clause that is in analogy with the last clause of iv.) is again the only reasonable one if we want identity to satisfy the identity axioms.

Let us define entailment with the help of the L4 ordering relation again.

**Definition 8** *(Entailment in partial predicate logic) Let* $\Gamma$ *and* $\Delta$ *be sets of formulae.* $\Gamma$ entails $\Delta$, $\Gamma \models \Delta$, *if*

$$\bigcap_{\varphi \in \Gamma} |\varphi|^{M,a} \subseteq \bigcup_{\psi \in \Delta} |\psi|^{M,a}$$

*for all models* $M$ *and assignments* $a$ *for* $M$.

**What are this logic's properties?**
We can show it has many nice properties through an embedding of partial predicate logic into ordinary predicate logic. With its help we can get some important theorems for free.

The following definition gives the syntactic part.

**Definition 9** *With each n-ary relation symbol* $R$ *of our language* $L$ *we associate two n-ary symbols* $R^+$ *and* $R^-$. *Let* $L^\dagger$ *be the language consisting of (a) all relation symbols* $R^+$ *and* $R^-$ *associated with some* $R \in L$, *(b) all individual constants in* $L$ *and (c) two new zero place relation symbols* $p^+$ *and* $p^{Đ}$. *If* $\varphi$ *is a formula then we write* $\pm\varphi$ *for the result of simultaneously substituting each* $R^+$ *for its associated* $R^-$ *and each* $R^-$ *for its associated* $R^+$ *in* $\varphi$ *(*$p^+$ *and* $p^-$ *are meant to be included here). With each formula* $\varphi$ *of partial predicate logic we associate a formula* $\varphi^\dagger$ *of ordinary predicate logic with the help of the following clauses.*

*i.* $(Rt_1 \ldots t_n)^\dagger = R^+ t_1 \ldots t_n$;

*ii.* $(t_1 = t_2)^\dagger = t_1 = t_2$;

*iii.* $(\#)^\dagger = p^+$;
$(\star)^\dagger = p^-$;

*iv.* $(\neg\varphi)^\dagger = \neg \pm \varphi^\dagger$;
$(\varphi \wedge \psi)^\dagger = \varphi^\dagger \wedge \psi^\dagger$;
$(\varphi = \psi)^\dagger = (\varphi^\dagger \leftrightarrow \psi^\dagger) \wedge \pm(\varphi^\dagger \leftrightarrow \psi^\dagger)$;

*v.* $(\forall x \varphi)^\dagger = \forall x \varphi^\dagger$.

The following lemma holds.

**Embedding Lemma**. *Let* $M_4 = \langle D, I^+, I^- \rangle$ *be a model for partial predicate logic and let* $M_2 = \langle D, I_2 \rangle$ *be a model for standard predicate logic such that:*

*(i.)* $I_2(c) = I^+(c)$ *for each individual constant* $c \in L$*;*

*(ii.)* $I_2(p^+) = 1$, $I_2(p^-) = 0$*;*

*(iii.)* $I_2(R^+) = I^+(R)$ *and* $I_2(R^-) = D_n - I^-(R)$ *for all n-ary relation symbols R.*

*Then the following two equivalences hold for each assignment a :*

$M_4 \models \varphi[a]$ *iff* $M_2 \models \varphi^\dagger[a]$

$M_4 \mathrel{=\!\!|} \varphi[a]$ *iff* $M_2 \models \neg \pm \varphi^\dagger[a]$.

The proof of this lemma is a straightforward induction on the complexity of $\varphi$ which we leave to the reader. From the Embedding Lemma the next theorem follows.

**Theorem 3 (Embedding Theorem)** *Let* $\Gamma$ *and* $\Delta$ *be sets of sentences of partial predicate logic. Define* $\Gamma^\dagger$ *to be the set* $\{\varphi^\dagger \mid \varphi \in \Gamma\}$ *and similarly define* $\Delta^\dagger$ *to be* $\{\psi^\dagger \mid \psi \in \Delta\}$*. Write* $\models_2$ *for the relation of entailment in predicate logic. Then:*

$$\Gamma \models \Delta \text{ iff } \Gamma^\dagger, (p^+ \wedge \neg p^-) \vee (p^- \wedge \neg p^+) \models_2 \Delta^\dagger.$$

The proofs of this theorem and of its corollaries—see below—are in the Appendix as usual.

**A Rich Harvest of Corollaries:**
Let us call $M$ a *model of* a theory $\Sigma$ if $M \models \sigma$ for every $\sigma \in \Sigma$. We have:

**Corollary.** (Compactness Theorem for partial predicate logic) *If every finite subset of some theory has a model then that theory has a model.*

**Corollary.** (Löwenheim-Skolem Theorem for partial predicate logic) *If a theory has an infinite model then it has a countably infinite model.*

**Corollary.** *There is a recursive axiomatization of partial predicate logic.*

**Corollary.** *Let* $\Gamma$ *and* $\Delta$ *be sets of sentences of partial predicate logic that are built up from* $\neg$, $\wedge$, $\top$, $\bot$, $=$ *and* $\forall$*. Then* $\Gamma \models \Delta, \star$ *iff* $\Gamma, \star \models \Delta$.

This last corollary is of some practical value since it follows that in many cases it will be possible to see that some entailment holds after doing only half of the checking that is required in general.

- If some argument consists only of sentences built up from $\neg$, $\wedge$, $\top$, $\bot$, $=$ and $\forall$
- It suffices to prove either that truth is preserved from premises to conclusions or that falsity is preserved in the other direction.
- We need not prove both in these cases because either proposition implies the other.

We opened this chapter questioning one of the central principles of classical logic, the principle that a sentence is true if and only if it is not false. The answer is: surprisingly little. There is a simple function embedding the former logic into the latter and we can use this embedding in many cases to reduce questions about four-valued predicate logic to questions about standard logic.