Karttunen & Peters (1979) and Krahmer, Ch. 4

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1 Karttunen & Peters 1979: Introduction

(1) Tom likes Alice.

(2) Asserted Meaning: Tom likes Alice.

(3) Presupposed Meaning:

   a. Tom is male.
   b. Alice is female.
   c. Tom and Alice are acquainted with one another.

K&P argue that this kind of presupposed meaning (conventional implicatures in their terms) is a part of the compositional semantics of (1). Each presupposition can be associated with the lexical semantics of one of the heads in the sentence:

(4) a. 'Tom' encodes (3a)
   b. 'Alice' encodes (3b)
   c. 'likes' encodes (3c)

If this is part of the compositional meaning, let's compose it just as we do the asserted meaning in (2).

The denotation of a head α is <**A**(α), **P**(α)>:

- **A**(α): the asserted meaning
- **P**(α): the presupposed meaning

This kind of system allows not only examples like (1), but also the following:

(5) Tom managed to complete the course.
   a. **A**(5): Tom completed the course.
   b. **P**(5): It was difficult for Tom to complete the course.

(6) Even TOM completed the course.
   a. **A**(6): Tom completed the course.

b. **P**(6):
   i. Existential Implicature: There is someone other than Tom who completed the course.
   ii. Scalar Implicature: Of the people who completed the course, Tom is the least likely to have completed the course.

This system can also account for how presuppositions are passed on in larger embedded structures as in (7):

(7) If the king of France is bald then baldness is hereditary.

2 'Tom likes Alice': The Mechanics of the System

Each head is associated with two meanings, the asserted as presupposed:

- The Denotation of 'Tom': <**A**(Tom), **P**(Tom)>:
  - **A**(Tom): $\lambda P_{et}.\ P(\text{Tom})$
  - **P**(Tom): $\lambda P_{et}.\ \text{male(Tom)}$
- The Denotation of 'Alice': <**A**(Alice), **P**(Alice)>:
  - **A**(Alice): $\lambda P_{et}.\ P(\text{Alice})$
  - **P**(Alice): $\lambda P_{et}.\ \text{male(Alice)}$
- The Denotation of 'like': <**A**(like), **P**(like)>:
  - **A**(like): $\lambda Q_{(et)t}\ \lambda x_e.\ Q(\lambda y_e.\ \text{like}(x, y))$
  - **P**(like): $\lambda Q_{(et)t}\ \lambda x_e.\ Q(\lambda y_e.\ \text{acquainted-with}(x, y))$

K&P modify Montague's semantic Rules to deal both with **A**meaning and **P**meaning.

- RULE 5 (composing objects with verbs)
  - $\mathbf{A}(\delta(\beta)) = \mathbf{A}(\delta)(\mathbf{A}(\beta))$
  - $\mathbf{P}(\delta(\beta)) = \lambda \text{x}_e[\mathbf{P}(\delta)(\mathbf{A}(\beta))(x) \wedge \exists P\ \mathbf{P}(\beta)(P)]$

$$
\begin{aligned}
\mathbf{A}(\text{likes Alice}) &= \mathbf{A}(\text{like})(\mathbf{A}(\text{Alice})) \\
&= [\lambda \text{Q}_{(et)t}\ \lambda \text{x}_e.\ \text{Q}(\lambda \text{y}_e.\ \text{like(x,y)})]\ (\lambda P_{et}.\ \mathbf{P}(\text{Alice})) \\
&= \lambda \text{x}_e.\ \text{like(x,Alice)} \\
\mathbf{P}(\text{likes Alice}) &= \lambda \text{x}_e[\mathbf{P}(\text{likes})(\mathbf{A}(\text{Alice}))(\text{x}) \wedge \exists P_{et}\ \mathbf{P}(\text{Alice})(P)] \\
&= \lambda \text{x}_e[\ [\lambda \text{Q}_{(et)t}\ \lambda \text{x}_e.\ \text{Q}(\lambda \text{y}_e.\ \text{acquainted-with(x,y)})](\lambda P_{et}.\ P(\text{Alice}))(\text{x}) \wedge \\
&\quad \lambda P_{et}.\ \text{female(Alice)}\ (P)] \\
&= \lambda \text{x}_e.\ [\text{acquainted-with(x,Alice)} \wedge \text{female(Alice)}]
\end{aligned}
$$

- RULE 4 (composing subjects with verb phrases)

- $\mathbf{A}(\alpha(\delta)) = \mathbf{A}(\alpha)(\mathrm{A}(\delta))$
- $\mathbf{P}(\alpha(\delta)) = \mathbf{P}(\alpha)(\mathbf{A}(\delta)) \wedge \mathbf{A}(\alpha)(\mathbf{P}(\delta))$

$$\begin{array}{lcl}
\mathbf{A}(\text{Tom likes Alice}) & = & \mathbf{A}(\text{Tom})(\mathbf{A}(\text{likes Alice})) \\
& = & \lambda P_{et}.\ P(\text{Tom})[\lambda \mathrm{x}_e.\ \text{like(x,Alice)}] \\
& = & \text{like(Tom,Alice)} \\
\mathbf{P}(\text{Tom likes Alice}) & = & \mathbf{P}(\text{Tom})(\mathbf{A}(\text{likes Alice})) \wedge \mathbf{A}(\text{Tom})(\mathbf{P}(\text{likes Alice})) \\
& = & [\lambda P_{et}.\ \text{male(Tom)}](\lambda \mathrm{x}_e.\ \text{like(x,Alice)}) \wedge \\
& & [\lambda P_{et}.\ P(\text{Tom})](\lambda \mathrm{x}_e.\ [\text{acquainted-with(x,Alice)} \wedge \text{female(Alice)}]) \\
& = & \text{male(Tom)} \wedge \text{acquainted-with(Tom,Alice)} \wedge \text{female(Alice)}
\end{array}$$

Note: while to get the **A** meaning of a sentence we just compose the **A** meanings of its parts, we cannot so simply string **P** meanings together:

$$\begin{array}{lcl}
\mathbf{P}(\text{Tom likes Alice}) & = & \mathbf{P}(\text{Tom})(\mathbf{P}(\text{like})(\mathbf{P}(\text{Alice}))) \\
& = & [\lambda P_{et}.\ \text{male(Tom)}]\ [\lambda Q_{(et)t}\ \lambda \mathrm{x}_e.\ Q(\lambda \mathrm{y}_e.\ \text{acquainted-with(x,y)})] \\
& & (\lambda P_{et}.\ \text{female(Alice)})
\end{array}$$

This would not compose, and it would give a meaning of maleness being acquainted with femaleness.

The key is to flip one element to a **P** meaning at a time:

$$\begin{array}{lcl}
\mathbf{P}(\alpha(\delta(\beta))) & = & \mathbf{P}(\alpha)(\mathbf{A}(\delta)(\mathbf{A}(\beta))) \wedge \\
& = & \mathbf{A}(\alpha)(\mathbf{P}(\delta)(\mathbf{A}(\beta))) \wedge \\
& = & \mathbf{P}(\beta)
\end{array}$$

3 Manage

(8) Tom managed to complete the course.

In examples like (8), ‘manage’ in effect contributes nothing to the asserted meaning. The truth conditions for (8) are the same as they are for (9):

(9) Tom completed the course.

The A meaning of ‘manage’ just passes on the truth conditions of the rest of the sentence.

- $\mathbf{A}(\text{manage-to}) = \lambda P_{et}.P$

The **P** meaning of ‘manage’ implicates that it was difficult to do so:

- $\mathbf{P}(\text{manage-to}) = \lambda P_{et}.\ \lambda x_e.\ \neg\mathbf{A}(\text{easy})(P(x))$

We will return to examples like this in the Krahmer (6.1).

4 Even

K&P limit themselves to a small range of examples with 'even,' although they suggest that their analysis could be extended to account for other cases. In particular, they look at cases where a noun is focused, as in (10):

(10) Even TOM completed the course.

EVEN RULE: (see K&P p.52 for full rule) prefix 'even' to a focused NP and replace it with a variable in the scope sentence.

For (10), the result is something like the following:

(11) Even TOM [x completed the course].

Two factors are relevant for the presuppositions of 'even':

a. Focus: What NP is being focused? (here in all caps)
b. Scope: What is the scope of even?

'Even' has two presuppositions, one existential and one scalar. Here 'a' is the focused NP and [...x...] is the sentence in the scope of even.

- Existential: There are other x under consideration besides a, such that [...x...].
- Scalar: for all X under consideration besides a, the likelihood that [...x...] is greater than the likelihood that [...a...].

For (10) for example, we have:

- Existential: There are other x under consideration besides Tom such that x completed the course.
- Scalar: for all x under consideration besides Tom, the likelihood that x completed the course is greater than the likelihood that Tom completed the course.

P(even)(**A**(Tom), **A**(x completed the course):

- $\exists x_e.\ [*(x) \wedge \neg[x = \text{Tom}] \wedge \text{complete}(x,\text{the-course})] \wedge$
  $\forall x_e.\ [*(x) \wedge \neg[x = \text{Tom}] \rightarrow \mathbf{P}(\text{exceed})(\mathbf{P}(\text{likelihood})(\text{complete}(x,\text{the-course})),$
  $\mathbf{P}(\text{likelihood})(\text{complete}(\text{Tom},\text{the-course})))]$

*(x) restricts possible x to a relevant contextual set.

For (10), the scope is the entire sentence. Compare this with examples like (12):

(12) It's hard for me to believe that Tom can understand even SYNTACTIC STRUCTURES.

Here, we have a scope ambiguity. 'Even' could scope only over the embedded clause, as in (a) or over the entire sentence, as in (b):

a. Tom can understand x.

b. Its hard for me to believe that Tom can understand x

For (12), the ambiguity of scope leads to dramatically different scalar implicates:

a. For all x under consideration besides Syntactic Structures, the likelihood that Tom can understand x is greater than the likelihood that Tom can understand Syntactic Structures.

b. For all x under consideration besides Syntactic Structures, the likelihood that it is hard for me to believe that Tom can understand x is greater than the likelihood that it is hard for me to believe that Tom can understand Syntactic Structures.

Reading (a) suggests that Syntactic Structures if a difficult read. Reading (b) suggests that Syntactic Structures if very easy to read.

5 Conditionals & Filtering

K&P discuss conditionals, coordinations and disjunctions to see how presuppositions are passed up in larger structures. We will only focus here on conditionals. Before doing so, note the presuppositions of (13):

(13) The king of France is bald.

Example (14) presupposes that there is a king of France. With this in mind, compare the presupposition of (13) with the same clause embedded in a conditional as in (14) or (15):

(14) If the king of France is bald, then baldness is hereditary.

(15) If baldness is hereditary, the king of France is bald.

Intuitively, (14) still presupposes that there is a king of France. K&P argue more generally that in conditionals of the form

(16) if-φ-then-ψ

$\mathbf{P}(\varphi)$ is entailed. In other words:

(17) $\mathbf{A}$(if-φ-then-ψ) entails $\mathbf{P}(\varphi)$

As for (15), if-φ-then-ψ does not entail $\mathbf{P}(\psi)$. For $\mathbf{P}(\psi)$ to be true, $\mathbf{A}(\varphi)$ must be true. In other words:

(18) $\mathbf{A}(\varphi) \rightarrow \mathbf{P}(\psi)$

$\mathbf{P}$(if-φ-then-ψ) is the composite of (17) and (18):

(19) $\mathbf{P}$(if-φ-then-ψ) $= \mathbf{P}(\varphi) \wedge [\mathbf{A}(\varphi) \rightarrow \mathbf{P}(\psi)]$

K&P also note that to felicitously utter a sentence like (14), (where the antecedent is in the indicative) one must not know that the antecedent is false. Thus for them, the presuppositions of an indicative conditional is as in (20):

(20) $\mathbf{P}(\text{if-}\varphi\text{-then-}\psi) = [\neg K \neg \mathbf{A}(\varphi) \wedge \mathbf{P}(\varphi) \wedge [\mathbf{A}(\varphi) \rightarrow \mathbf{P}(\psi)]]$

The result then is that the presuppositions of the antecedent are passed along as presuppositions of the conditional. The presuppositions of the consequent is changed or filtered, so as to be a conditional presupposition, contingent on the truth of $\mathbf{A}(\varphi)$.

K&P also discuss one further situation, as in (21):

(21) If there is a king of France, the king of France is bald.

The consequent here presupposes that there is a king of France. K&P argue that this presupposition disappears from the conditional as a whole when the antecedent entails the presuppositions of the consequent. In other words, in example (21), $\mathbf{A}(\varphi) = \mathbf{P}(\psi)$, thus making the consequent of $[\mathbf{A}(\varphi) \rightarrow \mathbf{P}(\psi)]$ vacuous.

6 Krahmer: CH. 4 — Some Data

6.1 Presupps and Quantification (or, the "Binding Problem")

(22) Somebody managed to succeed George V on the throne of England.

- Presupposes that it was hard for somebody to succeed George V (as it would have been for most people).
- But it wasn't actually hard for George VI, the person who did—in no sense did he only "manage" to do it.
- Another presupposition failure—but KP'79 don't predict that it should fail easily.

(23) A fat man pushes his bicycle.

- Heim '83: for any choice of fat man, there must be some bike that he pushes.
- This, thus, presupposes that every fat man has a bike.

The problem for K&P in these examples is that assertion and presupposition are composed and calculated separately under their theory. Thus, quantification must be calculated separately for each condition as well.

6.2 Krahmer's Main Idea

- Partiality to the rescue!
- He draws on the notion from Strawson (1950):
  - π is a presupposition of φ if, and only if, whenever π is not true, φ is neither true nor false.

  - or, alternatively, whenever φ is either true for false, π is true.

- Importantly, π is the maximal presupposition; unlike K&P, Krahmer isn't really concerned with composition of elementary presuppositions.
- Basically, for any formula φ, we should be able to equate a presupposition π with the disjunction $\varphi \vee \neg\varphi$.
- This only makes sense in a partially valued logic, since $\varphi \vee \neg\varphi$ is otherwise a tautology.
- So, if we can associate presuppositions with this disjunction, we can eliminate the need for a separate composition, and accordingly, solve the binding problem.

7 Partial Predicate Logic (PPL)

7.1 Syntax

The syntax of PPL is just like normal Predicate Logic (PL), except for the following addition:

- If φ,π are formulae, then $\varphi_{<\pi>}$ is a formula, where π is an elementary (or potential) presupposition associated with φ.

7.2 Semantics

7.2.1 {Strong|Middle|Weak} Kleene Connectives

Krahmer uses a three valued logic with values T(rue), F(alse), and N(either True nor False). He introduces three sets of Kleene connectives for use with PPL:

Strong:

$\wedge$	T	F	N	$\rightarrow$	T	F	N	$\vee$	T	F	N		$\neg$
T	T	F	N	T	T	F	N	T	T	T	T	T	F
F	F	F	F	F	T	T	T	F	T	F	N	F	T
N	N	F	N	N	T	N	N	N	T	N	N	N	N

Middle:

$\dot{\wedge}$	T	F	N	$\dot{\rightarrow}$	T	F	N	$\dot{\vee}$	T	F	N		$\neg$
T	T	F	N	T	T	F	N	T	T	T	T	T	F
F	F	F	F	F	T	T	T	F	T	F	N	F	T
N	N	N	N	N	N	N	N	N	N	N	N	N	N

Weak:

$\ddot{\wedge}$	T	F	N	$\ddot{\rightarrow}$	T	F	N	$\ddot{\vee}$	T	F	N		$\neg$
T	T	F	N	T	T	F	N	T	T	T	N	T	F
F	F	F	N	F	T	T	N	F	T	F	N	F	T
N	N	N	N	N	N	N	N	N	N	N	N	N	N

These connectives basically define the three kinds of Kleene PPL. Notice that the main difference between each of them is how they deal with the Neither (true nor false) value.

The Middle Kleene system is asymetric. The right-hand conjunct is only considered when the left is True. Otherwise, the value of the left hand is what is important.

Both Weak and Middle Kleene can be defined in terms of the Strong Kleene.

Today, for reasons of time, we will focus on the Strong and Middle definitions[1].

- For the interpretations in Strong Kleene, see **Definition 4** on p. 97.
- For the interpretations in Middle Kleene, see **Fact 5** on p. 99.
- **Definition 7:** Middle Kleene connectives in Strong Kleene
  1. $\varphi \dot{\wedge} \psi = (\varphi \wedge \psi) \vee (\varphi \wedge \neg\varphi)$
  2. $\varphi \dot{\vee} \psi = (\varphi \wedge \psi) \wedge (\varphi \vee \neg\varphi)$
  3. $\varphi \dot{\rightarrow} \psi = (\varphi \rightarrow \psi) \wedge (\varphi \vee \neg\varphi)$
- Notice that in Definition 7, the crucial $\varphi \vee \neg\varphi$ that we want to associate with π appears in the definitions of Middle Kleene connectives.

7.2.2 Presuppositions in PPL

- We will also need a definition for presupposition (**Definition 10**). This is essentially the notion from Strawson:
  - φ presupposes π iff for all models M and assignments g:
    if M,g $\models \neg\varphi \vee \varphi$, then M,g $\models \pi$
- This still can't do away with the binding problem, though.
- Krahmer introduces translation operators from PPL into PL:
  - **TR**$^+$ maps a PPL formula φ onto a PL formula φ' which is true iff φ is True
  - **TR**$^-$ maps a PPL formula φ onto a PL formula φ' which is true whenever φ is False
- Using these, we can basically collapse presuppositional material and asserted material into a single PL formula.
- From **Definition 11**: Strong Kleene TR$^+$ and TR$^-$
  - $\mathbf{TR}^+(\varphi_{<\pi>}) = \mathbf{TR}^+(\pi) \wedge \mathbf{TR}^+(\varphi)$
    $\mathbf{TR}^-(\varphi_{<\pi>}) = \mathbf{TR}^+(\pi) \wedge \mathbf{TR}^-(\varphi)$
- **Fact 7**: From PPL to PL
  1. $[\![\mathrm{TR}^+(\varphi)]\!]^{PL}{}_M \Leftrightarrow [\![\varphi]\!]^+{}_M$

[1] As Krahmer discusses, very shortly, Weak Kleene definitions get us the right results for the wrong reasons, because they let all presuppositions project, which is not what we will want.

2. $[\![\mathrm{TR}^-(\varphi)]\!]^{PL}{}_M \Leftrightarrow [\![\varphi]\!]^-{}_M$

- Let PR(φ) be the strongest presupposition of φ, then PR(φ) is defined as $\mathrm{TR}^+(\varphi) \vee \mathrm{TR}^-(\varphi)$
- φ presupposes PR(φ), for any PPL formula φ
- **Fact 9**: Middle Kleene-based TR^+ and TR^-

  1. $\mathrm{TR}^+(\varphi \dot\wedge \psi)) = \mathrm{TR}^+(\varphi) \wedge \mathrm{TR}^+(\psi)$
     $\mathrm{TR}^-(\varphi \dot\wedge \psi)) = (\mathrm{TR}^-(\varphi) \vee \mathrm{TR}^-(\psi)) \wedge (\mathrm{TR}^+(\varphi) \vee \mathrm{TR}^-(\varphi))$
  2. $\mathrm{TR}^+(\varphi \dot\vee \psi)) = (\mathrm{TR}^+(\varphi) \vee \mathrm{TR}^+(\psi)) \wedge (\mathrm{TR}^+(\varphi) \vee \mathrm{TR}^-(\varphi))$
     $\mathrm{TR}^-(\varphi \dot\vee \psi)) = \mathrm{TR}^-(\varphi) \wedge \mathrm{TR}^-(\psi)$
  3. $\mathrm{TR}^+(\varphi \dot\rightarrow \psi)) = (\mathrm{TR}^-(\varphi) \vee \mathrm{TR}^+(\psi)) \wedge (\mathrm{TR}^+(\varphi) \vee \mathrm{TR}^-(\varphi))$
     $\mathrm{TR}^-(\varphi \dot\rightarrow \psi)) = \mathrm{TR}^+(\varphi) \wedge \mathrm{TR}^-(\psi)$

- Notice that the instances of $(\mathrm{TR}^+(\varphi) \vee \mathrm{TR}^-(\varphi))$ are equivalent to PR(φ). These are (conveniently) built into the middle Kleene definitions.
- Finally, **Fact 11** will come in useful:
  For all φ without elementary presuppositions: PR(φ) $\Leftrightarrow \top$

7.3 How this Works

Let's return to some familiar examples (see §5 above)—conditionals:

(24) a. If the king of France is bald, then baldness is hereditary. $(\delta_{<\pi>} \rightarrow_* \xi)$
b. If baldness is hereditary, then the king of France is bald. $(\xi \rightarrow_* \delta_{<\pi>})$
c. If France has a king, the king of France is bald. $(\pi \rightarrow_* \delta_{<\pi>})$

(25) Some shorthand:
a. δ represents the proposition that the King of France is bald.
b. ξ represents the proposition that baldness is hereditary.
c. π represents the proposition that there is a king of France

(24a) and (24b) both project the presupposition that France has a King, but (24c) doesn't intuitively seem to.

7.3.1 Strong Kleene

- Strong Kleene Propositional Connnectives (p. 105)
  - $\mathrm{PR}(\varphi \wedge \psi) \Leftrightarrow (\mathrm{PR}(\varphi) \vee \mathrm{TR}^+(\neg\psi)) \wedge (\mathrm{TR}^+(\neg\varphi) \vee \mathrm{PR}(\psi))$
  - $\mathrm{PR}(\varphi \vee \psi) \Leftrightarrow (\mathrm{PR}(\varphi) \vee \mathrm{TR}^+(\psi)) \wedge (\mathrm{TR}^+(\varphi) \vee \mathrm{PR}(\psi))$
  - $\mathrm{PR}(\varphi \rightarrow \psi) \Leftrightarrow (\mathrm{PR}(\varphi) \vee \mathrm{TR}^+(\neg\psi)) \wedge (\mathrm{TR}^+(\neg\varphi) \vee \mathrm{PR}(\psi))$

- Given the Strong Kleene definition for material implication, we can see if they make the right predictions about (24):
  - $\delta_{<\pi>} \rightarrow \xi$ presupposes $\pi \vee \xi$ (See the Appendix A.1)
  - $\xi \rightarrow \delta_{<\pi>}$ presupposes $\neg\xi \vee \pi$
  - $\pi \rightarrow \delta_{<\pi>}$ presupposes $\neg\pi \vee \pi$ (which is a tautology)
- Under Strong Kleene, we predict that (24a) the proposition that baldness is hereditary is presupposed, **or** that france has a king.
- For (24b), it presupposes that if baldness is hereditary, then there is a king of France.
- It predicts no presupposition for (24c), which is what we want.

7.3.2 Middle Kleene

- Middle Kleene Propositional Connnectives (p. 106)
  - $\mathrm{PR}(\varphi \dot{\wedge} \psi) \Leftrightarrow \mathrm{PR}(\varphi) \wedge (\mathrm{TR}^{+}(\neg\varphi) \vee \mathrm{PR}(\psi))$
  - $\mathrm{PR}(\varphi \dot{\vee} \psi) \Leftrightarrow \mathrm{PR}(\varphi) \wedge (\mathrm{TR}^{+}(\varphi) \vee \mathrm{PR}(\psi))$
  - $\mathrm{PR}(\varphi \dot{\rightarrow} \psi) \Leftrightarrow \mathrm{PR}(\varphi) \wedge (\mathrm{TR}^{+}(\neg\varphi) \vee \mathrm{PR}(\psi))$[2]
- Middle Kleene makes the following predictions about (24):
  - $\delta_{<\pi>} \dot{\rightarrow} \xi$ presupposes π (See the Appendix A.2)
  - $\xi \dot{\rightarrow} \delta_{<\pi>}$ presupposes $\neg\xi \vee \pi$
  - $\pi \dot{\rightarrow} \delta_{<\pi>}$ presupposes $\neg\pi \vee \pi$
- Under Middle Kleene, we predict that (24a) only the presupposition that france has a king (which is better than strong).
- It predicts no presupposition for (24c), which is still what we want.
- For (24b), there is still a weak presupposition that if baldness is hereditary, then there is a king of France.
- This basically matches up with K&P

8 Quantification

Recall (22), slightly truncated here:

(22) Somebody managed to succeed George V.

[2]Compare with (19)

We can roughly translate this as the following, where S represents "succeed George V", and where D represents "difficult to succeed George V":

(26) $\exists x(S(x)_{<D(x)>})$

Strong, Middle, and Weak Kleene definitions all have the following for existentials:

(27) $\text{PR}(\exists x\varphi) \Longleftrightarrow \exists x\text{TR}^{+}(\varphi) \vee \forall x\text{TR}^{-}(\varphi)$

We end up with:

(28) $\exists x(D(x) \wedge S(x)) \vee \forall x(D(x) \wedge \neg S(x))$ (See Appendix B)

In other words, either someone had difficulty succeeding George V but did so anyway, or everybody found it difficult, and nobody did.

- The first disjunct gives the conditions under which (22) is true.
- The second gives the conditions under which it is false.

Thus, the oddity of (22) is explained: both of these conditions are contradictory to how things actually were.

A Strong and Middle Kleene Conditionals

A.1 Strong Kleene

- $\text{PR}(\delta_{\pi} \rightarrow \xi) \Longleftrightarrow$
  $(\text{PR}(\delta_{\pi}) \vee \text{TR}^{+}(\xi)) \wedge (\text{TR}^{+}(\neg\delta_{\pi}) \vee \text{PR}(\xi)) \Longleftrightarrow$
  $((\text{TR}^{+}(\pi) \wedge \text{PR}(\delta) \vee \text{TR}^{+}(\xi)) \wedge (\text{TR}^{+}(\neg\delta_{\pi}) \vee \top \Longleftrightarrow$
  $((\text{TR}^{+}(\pi) \vee \text{TR}^{+}(\xi))$

A.2 Middle Kleene

- $\text{PR}(\delta_{\pi} \dot{\rightarrow} \xi) \Longleftrightarrow$
  $\text{PR}(\delta_{\pi}) \wedge (\text{TR}^{+}(\neg\delta_{\pi}) \vee \text{PR}(\xi)) \Longleftrightarrow$
  $(\text{TR}^{+}(\pi) \wedge \text{PR}(\delta)) \wedge (\text{TR}^{+}(\neg\delta_{\pi}) \vee \top \Longleftrightarrow$
  $\text{TR}^{+}(\pi)$

B Quantification

- $\exists x(S(x)_{<D(x)>})$
- $\text{PR}(\exists x(S(x)_{<D(x)>})) \Longleftrightarrow$
  $\exists x\text{TR}^{+}(S(x)_{<D(x)>}) \vee \forall x\text{TR}^{-}(S(x)_{<D(x)>}) \Longleftrightarrow$
  $\exists x(\text{TR}^{+}(D(x) \wedge \text{TR}^{+}(S(x))) \vee \forall x(\text{TR}^{+}(D(x)) \wedge \text{TR}^{-}(S(x)))$
  $\exists x(D(x) \wedge S(x)) \vee \forall x(D(x) \wedge \neg S(x))$