Handout 1: Quantification in First-Order Logic

Semantics C (Spring 2010)

The essence of quantification in classical (static) first-order logic (FOL) – or in dynamic predicate logic (DPL):

- *quantification is pointwise manipulation of variable assignments*

We'll see this by looking at the following example:

(1) Every^x woman saw a^y man that had a^z mustache.

(2) $\forall x(woman(x) \rightarrow \exists y(man(y) \wedge \exists z(mustache(z) \wedge have(y,z)) \wedge see(x,y)))$

1 FOL Syntax

(3) Basic expressions:

a. Terms: names (individual constants) $mary, john, \ldots$ and a denumerably infinite set of individual variables $\mathcal{V} = \{x, y, z, \ldots\}$

b. Predicates: one-place predicates $man, woman, \ldots$, two-place predicates $see, have, \ldots$, three-place predicates $give, send, \ldots$ etc.

(4) Atomic formulas:

a. If π is an n-place predicate and $\alpha_1, \ldots, \alpha_n$ are terms, then $\pi(\alpha_1, \ldots, \alpha_n)$ is a formula.

b. If α and β are terms, then $\alpha = \beta$ is a formula.

(5) Formulas (sentential connectives):

a. If ϕ is a formula, then $(\neg\phi)$ is a formula.

b. If ϕ and ψ are formulas, then $(\phi \wedge \psi)$ is a formula.

(6) Formulas (quantifiers):

a. If ϕ is a formula and v is a variable, then $(\exists v \phi)$ is a formula.

(7) Abbreviations (parentheses are omitted whenever they are contextually retrievable):

a. $(\phi \vee \psi) := (\neg(\neg\phi \wedge \neg\psi))$

b. $(\phi \rightarrow \psi) := (\neg(\phi \wedge \neg\psi))$

c. $(\phi \leftrightarrow \psi) := ((\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi))$

d. $(\forall v \phi) := (\neg\exists v(\neg\phi))$

2 FOL Semantics

(8) Models and assignments:

a. A model $\mathfrak{M}$ for FOL is a pair $\left\langle \boldsymbol{D}^{\mathfrak{M}}, \boldsymbol{I}^{\mathfrak{M}} \right\rangle$, where $\boldsymbol{D}$ is the domain of individuals in $\mathfrak{M}$ and $\boldsymbol{I}$ is a function that assigns an individual in $\boldsymbol{D}$ to every name and a subset of D^n to every n-place predicate (the superscript $^{\mathfrak{M}}$ on $\boldsymbol{D}^{\mathfrak{M}}$ and $\boldsymbol{I}^{\mathfrak{M}}$ is omitted whenever it is contextually retrievable).

b. An $\mathfrak{M}$-assignment $g, h, \ldots$ is a (total) function from the set of variables $\mathcal{V}$ to the set of individuals $\boldsymbol{D}$.

(9) Abbreviations:

a. $\mathbb{T}$ and $\mathbb{F}$ stand for true and false, respectively.

b. $g[\upsilon]h :=$ g differs from h at most with respect to the value assigned to the variable υ.

Thus, $g[\upsilon]h$ holds iff, for any variable υ' different from υ, we have that $g(\upsilon') = h(\upsilon')$ – and, possibly, but not necessarily, $g(\upsilon) = h(\upsilon)$. Note that $g[\upsilon]h$ is an equivalence relation.

2.1 Standard Version

The definition of the interpretation function $[\![\cdot]\!]^{\mathfrak{M},g}$, i.e., $[\![\cdot]\!]^{\langle \boldsymbol{D},\boldsymbol{I}\rangle,g}$ – or $[\![\cdot]\!]^{g}$ for short:

(10) Basic expressions:

a. If α is a name and π is an n-place predicate, then $[\![\alpha]\!]^{\langle \boldsymbol{D},\boldsymbol{I}\rangle,g} = \boldsymbol{I}(\alpha)$ and $[\![\pi]\!]^{\langle \boldsymbol{D},\boldsymbol{I}\rangle,g} = \boldsymbol{I}(\pi)$.

b. If υ is a variable, then $[\![\upsilon]\!]^{\langle \boldsymbol{D},\boldsymbol{I}\rangle,g} = g(\upsilon)$.

(11) Atomic formulas:

a. If π is an n-place predicate and $\alpha_1, \ldots, \alpha_n$ are terms, then $[\![\pi(\alpha_1, \ldots, \alpha_n)]\!]^g = \mathbb{T}$ iff $\langle [\![\alpha_1]\!]^g, \ldots, [\![\alpha_n]\!]^g \rangle \in [\![\pi]\!]^g$.

b. If α and β are terms, then $[\![\alpha = \beta]\!]^g = \mathbb{T}$ iff $[\![\alpha]\!]^g = [\![\beta]\!]^g$.

(12) Formulas (sentential connectives):

a. $[\![\neg\phi]\!]^g = \mathbb{T}$ iff $[\![\phi]\!]^g = \mathbb{F}$.

b. $[\![\phi \wedge \psi]\!]^g = \mathbb{T}$ iff $[\![\phi]\!]^g = \mathbb{T}$ and $[\![\psi]\!]^g = \mathbb{T}$.

(13) Formulas (quantifiers):

a. $[\![\exists\upsilon\phi]\!]^g = \mathbb{T}$ iff there is an assignment h such that $g[\upsilon]h$ and $[\![\phi]\!]^h = \mathbb{T}$.

(14) Based on the abbreviations in (7) above, we derive the following:

a. $[\![\phi \vee \psi]\!]^g = \mathbb{T}$ iff $[\![\phi]\!]^g = \mathbb{T}$ or $[\![\psi]\!]^g = \mathbb{T}$.

b. $[\![\phi \rightarrow \psi]\!]^g = \mathbb{T}$ iff, if $[\![\phi]\!]^g = \mathbb{T}$, then $[\![\psi]\!]^g = \mathbb{T}$
(equivalently: $[\![\phi \rightarrow \psi]\!]^g = \mathbb{T}$ iff $[\![\phi]\!]^g = \mathbb{F}$ or $[\![\psi]\!]^g = \mathbb{T}$).

c. $[\![\phi \leftrightarrow \psi]\!]^g = \mathbb{T}$ iff $[\![\phi]\!]^g = [\![\psi]\!]^g$.

d. $[\![\forall\upsilon\phi]\!]^g = \mathbb{T}$ iff any assignment h such that $g[\upsilon]h$ is such that $[\![\phi]\!]^h = \mathbb{T}$.

(15) Truth:

a. A formula ϕ is true in model $\mathfrak{M}$ iff $[\![\phi]\!]^{\mathfrak{M},g} = \mathbb{T}$ for any assignment g.

b. A formula ϕ is false in model $\mathfrak{M}$ iff $[\![\phi]\!]^{\mathfrak{M},g} = \mathbb{F}$ for any assignment g.

2.2 $\wp(G)$ Version

Note that the definition of truth in (15) above equates truth with the set of all variable assignments $G = \boldsymbol{D}^{\mathcal{V}}$. We can actually take the space of FOL denotations for formulas to be $\wp(G)$, with truth being G itself and falsity being $\emptyset$.

That is, we can take the denotations of formulas in FOL to be sets of variable assignments – and we can define an interpretation function $[\![\cdot]\!]^{\mathfrak{M}}$ from FOL formulas to $\wp(G)$ that is not parametrized by variable assignments, but only by the model $\mathfrak{M}$. This definition will be parallel to the definition of the interpretation function $[\![\cdot]\!]^{\mathfrak{M},g}$ above.

The definition of the interpretation function $[\![\cdot]\!]^{\mathfrak{M}}$, i.e., $[\![\cdot]\!]^{\langle \boldsymbol{D},\boldsymbol{I}\rangle}$ – or $[\![\cdot]\!]$ for short:

(16) Abbreviation: for any term α and any assignment g, let $g/\boldsymbol{I}(\alpha) := \begin{cases} g(\alpha), & \text{if } \alpha \text{ is a variable} \\ \boldsymbol{I}(\alpha), & \text{if } \alpha \text{ is a name} \end{cases}$

(17) Atomic formulas:

a. If π is an n-place predicate and $\alpha_1, \ldots, \alpha_n$ are terms, then $[\![\pi(\alpha_1, \ldots, \alpha_n)]\!] = \{g : \langle g/\boldsymbol{I}(\alpha_1), \ldots, g/\boldsymbol{I}(\alpha_n)\rangle \in \boldsymbol{I}(\pi)\}$.

b. If α and β are terms, then $[\![\alpha = \beta]\!] = \{g : g/\boldsymbol{I}(\alpha) = g/\boldsymbol{I}(\beta)\}$.

(18) Formulas (sentential connectives):

a. $[\![\neg\phi]\!] = \{g : g \notin [\![\phi]\!]\} = G \backslash [\![\phi]\!]$

b. $[\![\phi \wedge \psi]\!] = [\![\phi]\!] \cap [\![\psi]\!]$.

(19) Formulas (quantifiers):

a. $[\![\exists \upsilon \phi]\!] = \{g : \text{there is an } h \text{ such that } g[\upsilon]h \text{ and } h \in [\![\phi]\!]\} = \{g : (\{h : g[\upsilon]h\} \cap [\![\phi]\!]) \neq \emptyset\}$

(20) Based on the abbreviations in (7) above, we derive the following:

a. $[\![\phi \vee \psi]\!] = [\![\phi]\!] \cup [\![\psi]\!]$

b. $[\![\phi \rightarrow \psi]\!] = \{g : g \notin [\![\phi]\!] \backslash [\![\psi]\!]\} = G \backslash ([\![\phi]\!] \backslash [\![\psi]\!]) = (G \backslash [\![\phi]\!]) \cup [\![\psi]\!]$

c. $[\![\phi \leftrightarrow \psi]\!] = ([\![\phi]\!] \cap [\![\psi]\!]) \cup ((G \backslash [\![\phi]\!]) \cap (G \backslash [\![\psi]\!])) = ([\![\phi]\!] \cap [\![\psi]\!]) \cup (G \backslash ([\![\phi]\!] \cup [\![\psi]\!]))$

d. $[\![\forall \upsilon \phi]\!] = \{g : \text{any } h \text{ such that } g[\upsilon]h \text{ is such that } h \in [\![\phi]\!]\} = \{g : \{h : g[\upsilon]h\} \subseteq [\![\phi]\!]\}$

(21) Truth:

a. A formula ϕ is true in model $\mathfrak{M}$ iff $[\![\phi]\!] = G$.

b. A formula ϕ is false in model $\mathfrak{M}$ iff $[\![\phi]\!] = \emptyset$.

Note the emerging parallel between FOL semantics and the Kripke semantics for modal logic.

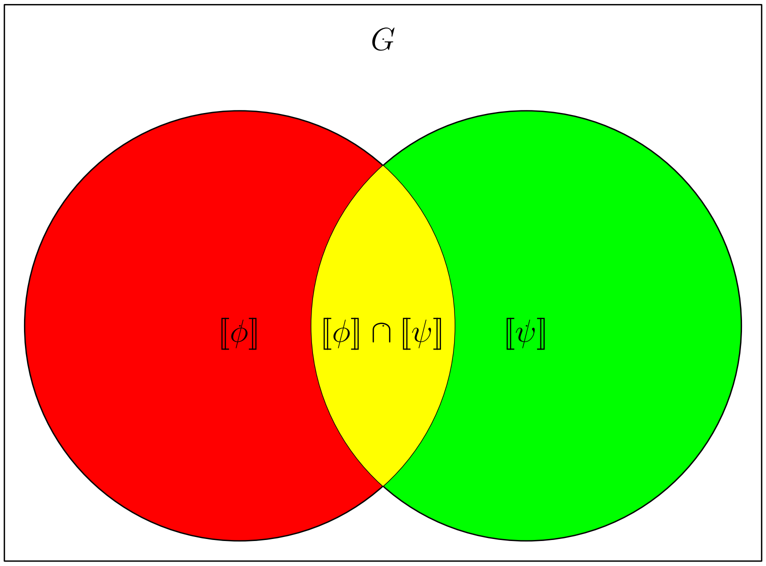
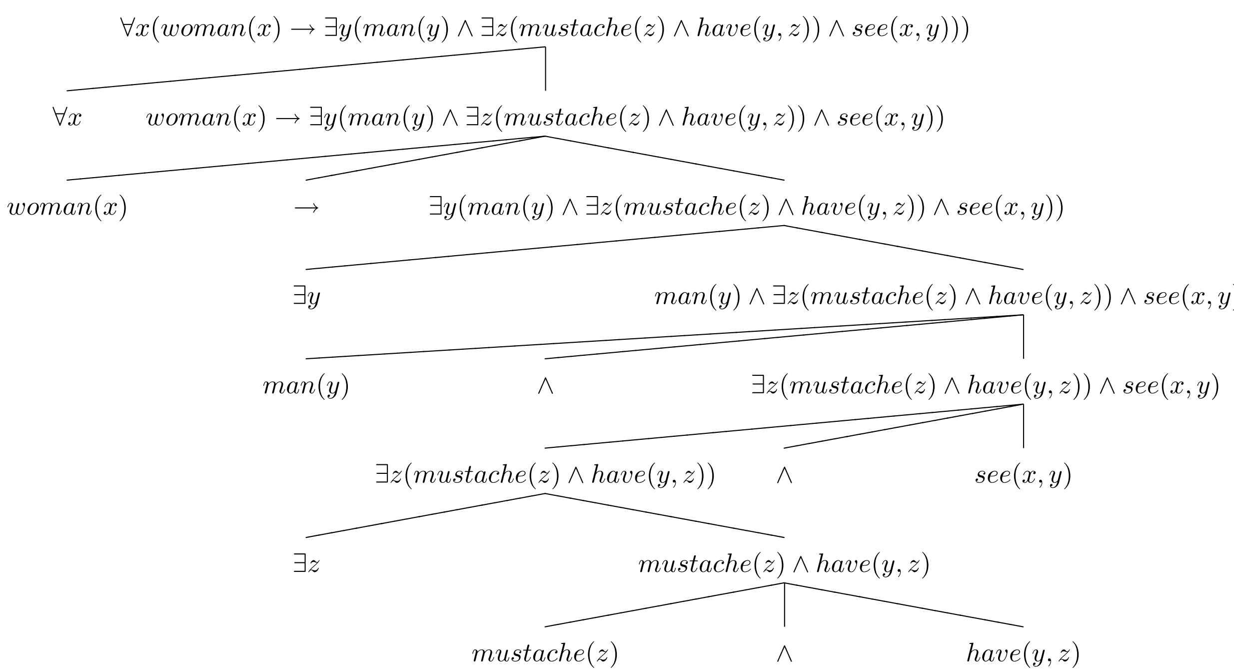


Figure 1: G, $[\![\phi]\!]$ and $[\![\psi]\!]$

3 Back to Our Example

Going back to the formula in (2) above, we have the syntactic structure in (22) below.

(22)



Using the standard definition of FOL semantics, we have:

(23) $[\![\forall x(woman(x) \rightarrow \exists y(man(y) \wedge \exists z(mustache(z) \wedge have(y,z)) \wedge see(x,y)))]\!]^g = \mathbb{T}$ iff

a. any h such that $g[x]h$ is such that
$[\![woman(x) \rightarrow \exists y(man(y) \wedge \exists z(mustache(z) \wedge have(y,z)) \wedge see(x,y))]\!]^h = \mathbb{T}$ iff

b. any h such that $g[x]h$ is such that, if $[\![woman(x)]\!]^h = \mathbb{T}$, then
$[\![\exists y(man(y) \wedge \exists z(mustache(z) \wedge have(y,z)) \wedge see(x,y))]\!]^h = \mathbb{T}$ iff

c. any h such that $g[x]h$ and $h(x) \in \boldsymbol{I}(woman)$ is such that
$[\![\exists y(man(y) \wedge \exists z(mustache(z) \wedge have(y,z)) \wedge see(x,y))]\!]^h = \mathbb{T}$ iff

d. any h such that $g[x]h$ and $h(x) \in \boldsymbol{I}(woman)$ is such that
there is an i such that $h[y]i$ and $[\![man(y) \wedge \exists z(mustache(z) \wedge have(y,z)) \wedge see(x,y)]\!]^i = \mathbb{T}$ iff

e. any h such that $g[x]h$ and $h(x) \in \boldsymbol{I}(woman)$ is such that
there is an i such that $h[y]i$ and $[\![man(y)]\!]^i = \mathbb{T}$ and $[\![\exists z(mustache(z) \wedge have(y,z))]\!]^i = \mathbb{T}$ and $[\![see(x,y)]\!]^i = \mathbb{T}$ iff

f. any h such that $g[x]h$ and $h(x) \in \boldsymbol{I}(woman)$ is such that
there is an i such that $h[y]i$ and $i(y) \in \boldsymbol{I}(man)$ and $[\![\exists z(mustache(z) \wedge have(y,z))]\!]^i = \mathbb{T}$ and $\langle i(x), i(y)\rangle \in \boldsymbol{I}(see)$ iff

g. any h such that $g[x]h$ and $h(x) \in \boldsymbol{I}(woman)$ is such that
there is an i such that $h[y]i$ and $i(y) \in \boldsymbol{I}(man)$ and
there is a j such that $i[z]j$ and $[\![mustache(z) \wedge have(y,z)]\!]^j = \mathbb{T}$
and $\langle i(x), i(y)\rangle \in \boldsymbol{I}(see)$ iff

h. any h such that $g[x]h$ and $h(x) \in \boldsymbol{I}(woman)$ is such that
there is an i such that $h[y]i$ and $i(y) \in \boldsymbol{I}(man)$ and
there is a j such that $i[z]j$ and $j(z) \in \boldsymbol{I}(mustache)$ and $\langle j(y), j(z)\rangle \in \boldsymbol{I}(have)$
and $\langle i(x), i(y)\rangle \in \boldsymbol{I}(see)$ iff

any woman $\mathbb{X}$ is such that there is a man $\mathbb{Y}$ that she saw and that had a mustache $\mathbb{Z}$.

Note that:

- $h(x) = i(x) = j(x)$
- $i(y) = j(y)$

4 Quantification in FOL

Quantifiers are interpreted as manipulating variable assignments in a pointwise (i.e., variable-wise) manner and passing the resulting assignments to the subformulas in their scope. Variable assignments are databases recording which variables are quantified over and how their values are restricted – and these databases 'mediate' / 'glue together' the interpretation of various parts of the formula.

For example, when we interpret $see(x,y)$, we interpret it relative to an assignment i that is already constrained by the previous quantifiers and subformulas (i.e., by the previous 'discourse') – since $i(x)$ is a woman and $i(y)$ is a man that has a mustache.

Thus, quantifiers ...

- manipulate variable assignments,
- place constraints on the set of resulting assignments (all/some/most/none/few of them have to be such that ...),
- pass them on to the subformulas in their scope (to 'subsequent discourse') and, finally,
- erase all the variable assignment manipulations after the subformulas in their scope are interpreted (e.g., $see(x,y)$ is not interpreted relative to the assignment j, but relative to the assignment i).

The last point is important – a translation procedure from English into FOL that is compositional down to clausal level (which is the most we can expect from first-order logic anyway) is not able to compositionally translate the following variation of the example in (1) above (antecedents are superscripted with the variable they introduce, while anaphors are subscripted with the variable they retrieve):

(24) Every^x woman saw a^y man that had a^z mustache and that was twisting it$_z$.

In particular, the following formula does not deliver the intuitively correct truth conditions:

(25) $\forall x(woman(x) \rightarrow \exists y(man(y) \wedge \exists z(mustache(z) \wedge have(y, z)) \wedge twist(y, z) \wedge see(x, y)))$

Why?

FOL differs from dynamic predicate logic (DPL) only with respect to the last point: we do not erase the variable assignment manipulations – a.k.a., updates – when we are done interpreting existential quantifiers. Thus, the DPL interpretation function needs to record not only the input variable assignment g, but also the output variable assignment h that is the result of the manipulations / updates contributed by quantifiers.

This single modification in the definition of the interpretation function shifts our perspective on meaning in several ways:

- static approaches (along FOL lines) equate the meaning of a sentence with its truth conditions, i.e., the circumstances in which a sentence is true or false
- dynamic approaches (along DPL lines) have a finer-grained conception of meaning: the meaning of a sentence is its context change potential, i.e., the way in which it changes / updates a (discourse) context