Chapter 6. Structured Nominal Reference: Quantificational Subordination

1. Introduction

The present chapter proposes an account of the contrast between the interpretations of the discourses in (1) and (2) below from Karttunen (1976).

1. **a.** Harvey courts a^u girl at every convention.
   b. She$_u$ is very pretty.
2. **a.** Harvey courts a^u girl at every convention.
   b. She$_u$ always comes to the banquet with him.
   [**c.** The$_u$ girl is usually also very pretty.]

The initial sentence (1a/2a) by itself is ambiguous between two readings (i.e. two quantifier scopings): it "can mean that, at every convention, there is some girl that Harvey courts or that there is some girl that Harvey courts at every convention. […] Harvey always courts the same girl […] [or] it may be a different girl each time" (Karttunen 1976: 377).

The contrast between the continuations in (1b) and (2b) is that the former allows only for the "same girl" reading of sentence (1a/2a), while the latter is also compatible with the "possibly different girls" reading.

Discourse (1) raises the following question: how can we capture the fact that a *singular anaphoric pronoun* in sentence (1b) can interact with and disambiguate *quantifier scopings* in sentence (1a)?

To see that it is indeed quantifier scopings that are disambiguated, substitute *exactly one^u girl* for *a^u girl* in sentence (1a); this will yield two truth-conditionally independent scopings: (**i**) *exactly one girl>>every convention*, which is true in a situation in which Harvey courts more than one girl per convention, but there is exactly one (e.g. Faye Dunaway) that he never fails to court, and (**ii**) *every convention>>exactly one girl*.

To see that number morphology on the pronoun *she* is indeed crucial, consider the discourse in (3) below, where the (preferred) relative scoping of *every convention* and *a girl* is the opposite of the one in discourse (1).

3. **a.** Harvey courts a^u girl at every convention. **b.** $They_u$ are very pretty.

Discourse (2) raises the following questions. First, why is it that adding an adverb of quantification, i.e. *always*/*usually*, makes both readings of sentence (2a) available?

Moreover, on the newly available reading of sentence (2a), i.e. the *every convention>>a girl* scoping, how can we capture the intuition that the singular pronoun *she* and the adverb *always* in sentence (2b) elaborate on the quantificational dependency between conventions and girls introduced in sentence (2a), i.e. how can we capture the intuition that we seem to have simultaneous *anaphora to two quantifier domains* and to the *quantificational dependency* between them?

The phenomenon instantiated by discourses (1) and (2) is subsumed under the more general label of *quantificational subordination* (see for example Heim 1990: 139, (2)), which covers a variety of phenomena involving interactions between generalized quantifiers and morphologically singular cross-sentential anaphora.

One of the main goals of this chapter is to show that the PCDRT system introduced in chapter **5** and motivated by mixed reading (weak & strong) donkey sentences receives independent empirical justification based on the phenomenon of quantificational subordination.

To account for quantificational subordination, we will only need to modify the definition of selective generalized quantification. As already remarked in section **3.5** of chapter **5**, there are two main strategies for the definition of generalized quantification in a dynamic system; the previous chapter explored one of them, namely the one that is closer to the DRT / FCS / DPL-style definition, while this chapter explores the other, formally more complex strategy, namely the one that is closer to van den Berg's definition of generalized quantification[1].

The chapter is structured as follows. Section **2** informally presents the PCDRT analysis of the Karttunen discourses in (1) and (2) above. Section **3** introduces and justifies the new definition of dynamic generalized quantification that enables us to account for quantificational subordination. Section **4** presents the formal PCDRT analysis of the Karttunen discourses based on the novel notion of dynamic quantification introduced in section **3**. Finally, section **6** briefly compares the PCDRT analysis of quantificational subordination with alternative accounts.

The appendix to the chapter introduces generalized selective distributivity, i.e. selective distributivity generalized to arbitrary types, and studies some of the formal properties of DRS-level selective distributivity.

The presentation of the PCDRT analysis of quantificational subordination in sections **3** and **4** repeats some of the basic notions and ideas introduced in the previous chapters. I hope that the resultant global redundancy is outweighed by the local improvement in readability.

2. Structured Anaphora to Quantifier Domains

This section shows semi-formally that the semantic framework proposed in the previous chapter (chapter **5**), i.e. PCDRT, enables us to account for discourses (1) and (2) above. In particular, the main proposal will be that compositionally assigning natural language expressions *finer-grained semantic values* (finer grained than the usual meanings assigned in static Montague semantics) enables us to capture the interaction between generalized quantifiers, singular pronouns and adverbs of quantification exhibited by the contrast between the interpretations of (1) and (2).

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[1] The fact that we are able to reformulate the two kidns of definitions of dynamic generalized quantification within the same type-logical system greatly facilitates their formal and empirical comparison, which (unfortunately) I must leave for a different occasion.

Just as in the previous chapter, the PCDRT semantic values are finer-grained in a very precise sense: the 'meta-types' **e** and **t** assigned to the denotations of the two kinds of 'saturated' expressions (names and sentences respectively) are assigned types that are complex than the corresponding types in static extensional Montague semantics. That is, the 'meta-type' **t** abbreviates $(st)((st)t)$, i.e. a sentence is interpreted as a DRS, and the 'meta-type' **e** abbreviates *se*, i.e. a name is interpreted as a dref. The denotation of a common noun like *girl* will still be of type **et** – see (4) below – and the denotation of a selective generalized determiner like every will still be of type (**et**)((**et**)**t**).

4. *girl* $\rightsquigarrow \lambda v_{\mathbf{e}}.\ [girl_{et}\{v\}]$, i.e. *girl* $\rightsquigarrow \lambda v_{\mathbf{e}}.\lambda I_{st}.\lambda J_{st}.\ I{=}J \wedge girl_{et}\{v\}J$

Accounting for *cross-sentential* phenomena in semantic terms (as opposed to purely / primarily pragmatic terms) requires some preliminary justification. First, the same kind of finer-grained semantic values are independently motivated by intra-sentential phenomena, as shown by the account of mixed weak & strong donkey sentences in the previous chapter.

Second, the phenomenon instantiated by discourses (1) and (2) is as much intra-sentential as it is cross-sentential. Note that there are four separate components that come together to yield the contrast in interpretation between (1) and (2): (**i**) the generalized quantifier *every convention*, (**ii**) the indefinite *a girl*, (**iii**) the singular number morphology on the pronoun *she* and (**iv**) the adverb of quantification *always/usually*. To derive the intuitively correct interpretations for (1) and (2), we have to attend to both the cross-sentential connections *a girl–she* and *every convention–always/usually* and the intra-sentential interactions *every convention–a girl* and *always–she*.

I conclude that an account of the contrast between (1) and (2) that involves a revamping of semantic values has sufficient initial plausibility to make its pursuit worthwhile.

The PCDRT plural info states enable us to encode discourse reference to both *quantifier domains*, i.e. *values*, and *quantificational dependencies*, i.e. *structure*, as shown in the matrix in (5) below.

5. Info State I	…	u		u'		…
i_1	…	x_1	(i.e. ui_1)	y_1	(i.e. $u'i_1$)	…
i_2	…	x_2	(i.e. ui_2)	y_2	(i.e. $u'i_2$)	…
i_3	…	x_3	(i.e. ui_3)	y_3	(i.e. $u'i_3$)	…
…	…	…		…		…

Quantifier domains (sets):
$\{x_1, x_2, x_3, \ldots\}$, $\{y_1, y_2, y_3, \ldots\}$

Quantifier dependencies (relations):
$\{\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \langle x_3, y_3\rangle, \ldots\}$

Just as before, the values are the sets of objects that are stored in the *columns* of the matrix, e.g. a dref u for individuals stores a set of individuals relative to a plural info state, since u is assigned an individual by each assignment (i.e. row). The structure is *distributively* encoded in the *rows* of the matrix: for each assignment / row in the plural info state, the individual assigned to a dref u by that assignment is structurally correlated with the individual assigned to some other dref u' by the same assignment.

Thus, plural info states enable us to pass information about both quantifier domains and quantificational dependencies across sentential / clausal barriers, which is exactly what we need to account for the interpretation of discourses (1) and (2). More precisely, we need the following two ingredients.

First, we need a suitable interpretation for *selective generalized determiners*, e.g. *every* in (1a/2a), which needs to do two things: (**i**) it stores in the plural info state the *restrictor and nuclear scope sets* of individuals that are related by the generalized determiner; (**ii**) it stores in the plural info state the *quantificational dependencies* between the individuals in the restrictor and / or nuclear scope set and any other quantifiers or indefinites in the restrictor or nuclear scope of the quantification.

For example, the indefinite *a girl* in (1a/2a) is in the nuclear scope of the *every*-quantification, while in the usual donkey examples (*Every farmer who owns a^u donkey beats it$_u$*), we have an indefinite in the restrictor of the quantification.

Given that a plural info state stores (**i**) sets of individuals and (**ii**) dependencies between such sets, both of them are available for subsequent anaphoric retrieval, e.g.

always and *she* in (2b) are simultaneously anaphoric to (**i**) *every convention* and *a girl* on the one hand and (**ii**) the dependency between conventions and girls on the other hand.

The second ingredient is a suitable interpretation of *singular number morphology* on pronouns, e.g. *she* in (1b) and (2b), that can interact with quantifiers and indefinites in the previous discourse, e.g. *every convention* and *a girl* in (1a/2a), and with quantifiers in the same sentence, e.g. the adverb *always* in (2b).

In particular, I will take the singular number morphology on *she* in (1b) to require that the set of individuals stored by the current plural info state relative to u be a singleton. This set of individuals is introduced by the indefinite *a girl* in (1a) – irrespective of whether the indefinite has wide or narrow scope relative to *every convention*. This is possible because we use plural info states, by means of which we store sets of individuals and pass them across sentential boundaries – we can thus constrain their cardinality by subsequent anaphoric elements like *she*.

If the indefinite *a girl* has narrow scope relative to *every convention*, the singleton requirement contributed by *she* applies to the set of all girls that are courted by Harvey at some convention or other. Requiring this set to be a singleton boils down to removing from consideration all the plural information states that would satisfy the narrow scope *every convention>>a girl*, but not the wide scope *a convention>>every girl*.

We therefore derive the intuition that, irrespective of which quantifier scoping we assume for sentence (1a), any plural info state that we obtain after a successful update with sentence (1b) is bound to satisfy the representation in which the indefinite *a*^{u} *girl* (or a quantifier like *exactly one*^{u} *girl*) takes wide scope.

In the case of discourse (2) however, the adverb of quantification *always* in (2b) – which is anaphoric to the nuclear scope set introduced by *every convention* in (2a) – can take scope over the singular pronoun *she*. In doing so, the adverb 'breaks' the plural info state containing all the conventions into smaller sub-states, each storing a particular convention. Then, the singleton requirement contributed by singular morphology on *she*$_{u}$ is enforced locally, relative to these sub-states, and not globally, relative to the whole plural info state. We therefore end up requiring that the courted girl is unique *per convention* and not across the board (the latter option being instantiated by discourse (1)).

The following section will introduce, explain and motivate the new definition of selective generalized quantification in PCDRT – and the corresponding (minor) adjustments of the meanings of indefinites, pronouns and definites.

3. Redefining Generalized Quantification

We turn now to the definition of *selective* generalized quantification in PCDRT.

3.1. Four Desiderata

The definition has to satisfy four desiderata, the first three of which are about anaphoric connections that can be established *internally*, within the generalized quantification (i.e. between antecedents in the restrictor and anaphors in the nuclear scope) and the last of which is about anaphora that can be established *externally* (i.e. between antecedents introduced by or within the quantification and anaphors that are outside the quantification).

First, we want our definition to be able to account for the fact that anaphoric connections between the restrictor and the nuclear scope of the quantification can in fact be established, i.e. we want to account for donkey anaphora.

Second, we want to account for such anaphoric connections while avoiding the proportion problem which *unselective* quantification (in the sense of Lewis 1975) runs into, i.e. we need the generalized determiner to relate *sets of individuals* (i.e. sets of objects of type *e*) and not sets of 'assignments' (i.e. sets of objects of type *s*).

Sentence (6) below provides a typical instance of the proportion problem: intuitively, (6) is false in a situation in which there are ten farmers, nine have a single donkey each that they do not beat, while the tenth has twenty donkeys and he is busy beating them all. But the unselective formalization of *most*-quantification as quantification over 'assignments' incorrectly predicts that (6) is true in the above situation because more than half of the <*farmer*, *donkey*> pairs (twenty out of twenty-nine) are such that the farmer beats the donkey.

6. Most farmers who own a^u donkey beat it_u.

The third desideratum is that the definition of selective generalized quantification should be compatible with both strong and weak donkey readings: we want to allow for the different interpretations associated with the donkey anaphora in (7) (Heim 1990) and (8) (Pelletier & Schubert 1989) below.

7. Most people that owned a^u slave also owned his_u offspring.
8. Every person who has a^u dime will put it_u in the meter.

Sentence (7) is interpreted as asserting that most slave-owners were such that, for *every* (strong reading) slave they owned, they also his offspring. Sentence (8) is interpreted as asserting that every dime-owner puts *some* (weak reading) dime of her/his in the meter.

We also need to allow for mixed weak & strong relative-clause sentences like the one in (9) below (i.e. the kind of sentence we have analyzed in chapter **5**). Sentence (9) is interpreted as asserting that, for any person that is a computer buyer and a credit card owner, for *every* computer s/he buys, s/he uses *some* credit card of her/his to pay for the computer.

9. Every person who buys a^u computer and has $a^{u'}$ credit card uses $it_{u'}$ to pay for it_u.

Thus, the first three, internal desiderata simply recapitulate the main points we have made in chapters **2** through **5** and they are only meant to ensure that the new definition of selective generalized quantification preserves all welcome the results we have previously obtained.

The fourth desideratum, however, is about the novel phenomenon of quantificational subordination we have introduced by means of the discourses in (1) and (2) above. These discourses indicate that selective generalized determiners need to make anaphoric information externally available, i.e. they need to introduce dref's for the *restrictor and nuclear scope sets of individuals* related by the generalized determiner that can be retrieved by subsequent anaphora. Furthermore, we also need to make available for anaphoric take-up the *quantificational dependencies* between different quantifiers and/or indefinites (see the discussion of discourse (2) in the previous section).

In more detail, generalized quantification supports anaphora to two sets: (**i**) the maximal set of individuals satisfying the restrictor DRS, i.e. the *restrictor set*, and (**ii**) the maximal set of individuals satisfying the restrictor and nuclear scope DRS's, i.e. the *nuclear scope set*[2]. Note that the latter set is the nuclear scope set that emerges as a consequence of the *conservativity* of natural language quantification – and, as Chierchia (1995) and van den Berg (1996a) (among others) observe, we need to build conservativity into the definition of dynamic quantification to account for the fact that the nuclear scope DRS can contain anaphors dependent on antecedents in the restrictor[3].

The discourse in (10) below exemplifies anaphora to nuclear scope sets: sentence (10b) is interpreted as asserting that the people that went to the beach are the students that left the party after 5 am (which, in addition, formed a majority of the students at the party).

10. **a.** Most^{u} students left the party after 5 am.
    b. They$_{u}$ went directly to the beach.

The discourses in (11) and (12) below exemplify anaphora to restrictor sets. Both examples involve determiners that are right downward monotonic, which strongly favor anaphora to restrictor sets as opposed to anaphora to nuclear scope sets.

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[2] Throughout the paper, I will ignore anaphora to complement sets, i.e. sets obtained by taking the complement of the nuclear scope relative to the restrictor, e.g. *Very few students were paying attention to the lecture. They were hungover.*

[3] Thus, in a sense, Chierchia (1995) and van den Berg (1996a) suggest that the conservativity universal proposed in Barwise & Cooper (1981) should be replaced by / derived from an 'anaphoric' universal that would have the form: the meanings of natural language determiners have to be such that they allow for anaphoric connections between the restrictor and nuclear scope of the quantification (I am indebted to Roger Schwarzschild, p.c., for making this observation clearer to me).

In a dynamic system, the 'anaphoric' universal boils down to the requirement that the nuclear scope update be interpreted relative to the info state that is the output of the restrictor update. And the two strategies of defining dynamic generalized quantification explored in chapter **5** and chapter **6** respectively are two different ways of implementing this requirement (see in particular the discussion in section **3.5** of chapter **5** and section **1** of chapter **6**, i.e. the present chapter).

11. **a.** No^u student left the party later than 10 pm.
 b. They$_u$ had classes early in the morning.
12. **a.** Very few^u people with a rich uncle inherit his fortune.
 b. Most of them$_u$ don't.

Consider (11) first: any successful update with a no^u quantification ensures that the nuclear scope set is empty and anaphora to it is therefore infelicitous; the only anaphora possible in (11) is anaphora to the restrictor set. The same thing happens in (12) albeit for a different reason: anaphora to the restrictor set is the only possible one because anaphora to the nuclear scope set would yield a contradiction, namely: most of the people with a rich uncle that inherit his fortune don't inherit his fortune.

Thus, a selective generalized determiner will receive a translation of the form provided in (13) below, which is in the spirit – but fairly far from the letter – of van den Berg (1996a) (see his definition (4.1) on p. 149).

13. $\mathrm{det}^{u,u'\sqsubseteq u} \rightsquigarrow \lambda P_{\mathbf{et}}.\lambda P'_{\mathbf{et}}.\ \mathbf{max}^{u}({}_{\langle u\rangle}(P(u)));\ \mathbf{max}^{u'\sqsubseteq u}({}_{\langle u'\rangle}(P'(u')));\ [\mathbf{DET}\{u, u'\}]$

The translation in (13) can be semi-formally paraphrased as follows.

First note that, as expected, $\mathrm{det}^{u,u'\sqsubseteq u}$ relates a restrictor dynamic property $P_{\mathbf{et}}$ and a nuclear scope dynamic property $P'_{\mathbf{et}}$. When these dynamic properties are applied to individual dref's, i.e. $P(u)$ and $P'(u')$, we obtain a restrictor DRS $P(u)$ and a nuclear scope DRS $P'(u')$ of type $\mathbf{t} := (st)((st)t)$.

Which brings us to the three sequenced updates in (13), namely $\mathbf{max}^{u}({}_{\langle u\rangle}(P(u)))$, $\mathbf{max}^{u'\sqsubseteq u}({}_{\langle u'\rangle}(P'(u')))$ and $[\mathbf{DET}\{u, u'\}]$. The first update is formed out of three distinct pieces, namely the restrictor DRS $P(u)$, the operator ${}_{\langle u\rangle}(\ldots)$ which takes scope over the restrictor DRS and, finally, the operator $\mathbf{max}^{u}(\ldots)$ that takes scope over everything else. The second update is formed out of the same basic pieces, i.e. the restrictor DRS $P'(u')$, the operator ${}_{\langle u'\rangle}(\ldots)$ and the operator $\mathbf{max}^{u'\sqsubseteq u}(\ldots)$. The last update is a test containing the static condition $\mathbf{DET}\{u, u'\}$ contributed by the particular determiner under consideration and which relates two individual dref's u and u'.

These are the individual dref's introduced by the generalized determiner, more exactly by the operators $\mathbf{max}^{u}(\dots)$ and $\mathbf{max}^{u'\sqsubseteq u}(\dots)$: they introduce the dref's u and u' respectively and u stores the restrictor set of individuals, while u' stores the nuclear scope set of individuals obtained via conservativity, which is encoded by the superscripted inclusion $u'\sqsubseteq u$.

The restrictor set u is the maximal set of individuals (maximality is contributed by $\mathbf{max}^{u}(\dots)$) such that, when we take each u-individual separately (distributivity is contributed by $_{\langle u\rangle}(\dots)$), this individual satisfies the restrictor dynamic property (i.e. $P(u)$).

The nuclear scope set u' is obtained in a similar way except for the requirement that it is the maximal structured subset of the restrictor set u (i.e. $\mathbf{max}^{u'\sqsubseteq u}(\dots)$). The notion of structured subset $u'\sqsubseteq u$ is introduced and discussed in the very next section.

We finally reach the third update, which tests that the restrictor set u and the nuclear scope set u' stand in the relation denoted by the corresponding static determiner **DET** (i.e. $\mathbf{DET}\{u, u'\}$).

As already mentioned, the three updates in (13) are sequenced, i.e. dynamically conjoined. Recall that dynamic conjunction ';' is interpreted as relation composition, as shown in (14) below.

14. $D_1; D_2 := \lambda I_{st}.\lambda J_{st}.\ \exists H_{st}(D_1IH \wedge D_2HJ)$ [4],

where D_1 and D_2 are DRS's of type $\mathbf{t} := (st)((st)t)$.

The remainder of this section is dedicated to formally spelling out the meaning of generalized determiners in (13) above and, also, the PCDRT meanings for indefinite articles and pronouns.

We will need: (**i**) two operators over plural info states, namely a *selective maximization* operator $\mathbf{max}^{u}(\dots)$ and a *selective distributivity* operator $_{\langle u\rangle}(\dots)$, which will

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[4] Also, recall the difference between dynamic conjunction ';', which is an abbreviation, and the official, classical static conjunction '$\wedge$'.

enable us to define updates of the form $\mathbf{max}^{u}(_{\langle u\rangle}(\ldots))$ and (**ii**) a notion of *structured subset* between two sets of individuals that requires the subset to preserve the quantificational dependencies, i.e. the structure, associated with the individuals in the superset – which will enable us to define $u'\sqsubseteq u$ and, thereby, updates of the form $\mathbf{max}^{u'\sqsubseteq u}(\ldots)$.

3.2. Structured Inclusion

Let us start with the notion of structured subset. Recall that plural info states store both values (quantifier domains) – in the columns of the matrix – and structure (quantifier dependencies) – in the rows of the matrix. We can therefore define two different notions of inclusion: one that takes into account only values, i.e. value inclusion, and one that takes into account both values and structure, i.e. structured inclusion. Let us examine them in turn.

Requiring a dref u_3 to simply be a *value* subset of another dref u_1 relative to an info state I is defined as shown in (15) below. For example, the info state I in (16) satisfies the condition $u_3\subseteq u_1$ because $u_3I=\{x_1, x_2, x_3\}\subseteq u_1I=\{x_1, x_2, x_3, x_4\}$.

15. $u_3\subseteq u_1 := \lambda I_{st}.\ u_3I\subseteq u_1I$

16. Info State I

	u_1	u_2	u_3
i_1	x_1	y_1	x_1
i_2	x_2	y_2	x_3
i_3	x_3	y_3	x_1
i_4	x_4	y_4	x_2

As the info state I in (16) shows, value inclusion disregards structure completely: the correlation / dependency between the u_1-individuals and the u_2-individuals, i.e. the relation $\{<x_1, y_1>, <x_2, y_2>, <x_3, y_3>, <x_4, y_4>\}$, is lost in going from the u_1-superset to the u_3-subset: as far as u_3 and u_2 are concerned, x_1 is still correlated with y_1, but it is now also correlated with y_3; moreover, x_2 is now correlated with y_4 and x_3 with y_2.

If we were to use the notion of value subset in (15) to define dynamic generalized quantification, we would make incorrect predictions. To see this, consider the discourse in (17) below, where u_1 stores the set of conventions[5] and u_2 stores the set of corresponding girls. Furthermore, assume that *every*^{u_1} *convention* takes scope over *a*^{u_2} *girl* and that the correlation between u_1-conventions and courted u_2-girls is the one represented in (16) above.

17. **a.** Harvey courts a^{u_2} girl at every^{u_1} convention.

 b. She$_{u_2}$ usually$^{u_3 \subseteq u_1}$ comes to the banquet with him.

Intuitively, the adverb *usually* is anaphoric to the set of conventions and sentence (17b) is interpreted as asserting that at most conventions, the girl courted by Harvey *at that convention* comes to the banquet with him. The dref u_3 in (16) above does store most conventions (three out of four), but it does not preserve the correlation between conventions and girls established in sentence (17a).

Note that a similarly incorrect result is achieved for donkey sentences like the one in (18) below: the restrictor of the quantification introduces a dependency between all the donkey-owning u_1-farmers and the u_2-donkeys that they own; the nuclear scope set u_3 needs to contain most u_1-farmers, but in such a way that the correlated u_2-donkeys remain the same. That is, the nuclear scope set contains a *most*-subset of donkey owning farmers that beat *their respective donkey(s)*. The info state in (16) above and the notion of value-only inclusion in (15) are yet again inadequate.

18. Most$^{u_1, u_3 \subseteq u_1}$ farmers who own a^{u_2} donkey beat it$_{u_2}$.

Thus, to capture the intra-sentential and cross-sentential interaction between anaphora and quantification, we need a notion of *structured inclusion*, i.e. a notion of *value inclusion* that *preserves structure*. That is, the only way to go from a superset to a subset should be by *discarding rows in the matrix*: in this way, we are guaranteed that the subset will contain *only* the dependencies associated with the superset (but not necessarily *all* dependencies – see below).

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[5] Note that, in the case of a successful *every*-quantification, the restrictor and the nuclear scope sets end up being identical (both with respect to value and with respect to structure – for more details, see (65) below and its discussion), so, for simplicity, I conflate them into dref u_1.

Following van den Berg (1996a), I will introduce a dummy / exception individual # that I will use as a tag for the rows in the matrix that should be discarded in order to obtain a structured subset *u'* of a superset *u* – as shown by the matrix in (20) below. The formal definition is provided in (19).

19. $u_3 \Subset u_1 := \lambda I_{st}.\ \forall i_s \in I(u_3 i = u_1 i \vee u_3 i = \#)$

20. Info State I	u_1	u_2	u_3
i_1	x_1	y_1	x_1
i_2	x_2	y_2	x_2
i_3	x_3	y_3	#
i_4	x_4	y_4	x_4

Unlike van den Berg (1996a), I will not take the introduction of the dummy individual # to require us to make the underlying logic partial, i.e. I will not assigned the undefined truth-value to a lexical relation that takes the dummy individual # as an argument, e.g. *girl*(*#*) or *courted_at*(#, x_1). Instead, I will take such lexical relations to simply be false[6,7], which will allow us to keep the underlying type logic classical. The fact that the dummy individual # always yields falsity (as opposed to always yielding truth) is meant to ensure that we do not introduce # as the default value of a dref that vacuously satisfies any lexical relation.

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[6] Conflating undefinedness and falsity in this way is a well-known 'technique' in the presupposition literature: a Fregean / Strawsonian analysis of definite descriptions distinguishes between what such descriptions contribute to the asserted content and what they contribute to the presupposed content associated with any sentence in which they occur. In contrast, the Russellian analysis of definite descriptions takes everything to be asserted, i.e. it conflates what is asserted and what is presupposed according to the Fregean / Strawsonian analysis. Therefore, if the presupposed content is not true, the Russellian will have falsity whenever the Fregean / Strawsonian will have undefinedness.

While this conflation seems to be counter-intuitive and ultimately incorrect in the case of presupposition, it does not seem to be so in the case of structured inclusion. At this point, I cannot see any persuasive argument (empirical or otherwise) for a formally unified treatment of structured inclusion and presupposition (albeit van den Berg seems to occasionally suggest the contrary, see for example van den Berg 1994: 11, fn. 9), so I will work with the simplest possible system that can model structured inclusion.

[7] We ensure that any lexical relation R of arity n (i.e. of type $e^n t$, defined recursively as in Muskens 1996: 157-158, i.e. as $e^0 t := t$ and $e^{m+1} t := e(e^m t)$) yields falsity whenever # is one of its arguments by letting $R \subseteq (\mathbf{D}_e^{\mathbf{M}} \setminus \{\#\})^n$.

At the same time, requiring the dummy individual # to falsify any lexical relation makes it necessary for us to define lexical relations in PCDRT as shown in (22) below. That is, atomic conditions discard / ignore the dummy party of the plural info state, i.e. $I_{u_1=\#} \cup \ldots \cup I_{u_n=\#}$, and are interpreted only relative to the non-discarded part of the plural info state, i.e. $I_{u_1\neq\#,\ldots,u_n\neq\#}$. Note also that they are interpreted distributively relative to this non-discarded part, i.e. we universally quantify over every 'assignment' *i* in $I_{u_1\neq\#,\ldots,u_n\neq\#}$.

21. $I_{u_1\neq\#,\ldots,u_n\neq\#} := \{i_s\in I\text{: } u_1i\neq\# \wedge \ldots \wedge u_ni\neq\#\}$

22. $R\{u_1, \ldots, u_n\} := \lambda I_{st}.\ I_{u_1\neq\#,\ldots,u_n\neq\#} \neq \emptyset \wedge \forall i_s\in I_{u_1\neq\#,\ldots,u_n\neq\#}(R(u_1i, \ldots, u_ni))$

Discarding the 'dummy' part of the info state when we evaluate the condition (as shown in (22) above) is crucial: if we were to interpret conditions relative to the entire plural info state, the condition would very often be false because the dummy individual # yields falsity – and we would not be able to allow for output info states like the one in (20) above, which we need to define dynamic quantification. Finally, the non-emptiness requirement enforced by the first conjunct in (22) rules out the degenerate cases in which a plural info state vacuously satisfies an atomic condition by being entirely 'dummy'.

Let us return to the notion of structured inclusion needed for dynamic quantification. Note that the notion of structured inclusion $\Subset$ defined in (19) above ensures that the subset inherits *only* the superset structure – but we also need it to inherit *all* the superset structure, which we achieve by means of the definition in (23) below.

23. $u'\sqsubseteq u := \lambda I_{st}.\ (u'\Subset u)I \wedge \forall i_s\in I(ui\in u'I_{u'\neq\#} \rightarrow ui=u'i)$

To see that we need the second conjunct in (23), consider again the donkey sentence in (7) above, i.e. *Most people that owned a*^{u} *slave also owned his*$_u$ *offspring*. This sentence is interpreted as talking about *every* slave owned by any given person – therefore, the nuclear scope set, which needs to be a *most*-subset of the restrictor set, needs to inherit *all* the superset structure, i.e., for any slave owner in the nuclear scope set, we need to associate with her/him *every* slave (and his offspring) that s/he owned.

3.3. Maximization, Distributivity and Selective Quantification

We turn now to the definition of the maximization and distributivity operators $\mathbf{max}^{u}$ and $\mathbf{dist}_{u}$, which are defined in the spirit – but not the letter – of the corresponding operators in van den Berg (1996a). Selective maximization plus selective distributivity[8] enable us to dynamize λ-abstraction over both *values*, i.e. individuals, and *structure*, i.e. the quantificational dependencies associated with the individuals. We will consequently be able to extract and store the restrictor and nuclear scope structured sets needed to define dynamic generalized quantification.

To see that we need maximization over both values and structure, consider the discourse in (24) below. Sentence (24b) elaborates on the relation between students and cakes introduced by the first sentence. Note that this relation is the Cartesian product of the set of students and the set of cakes, i.e. we want to introduce the set of *all* students, the set of *all* cakes and the *maximal relation / structure* associating the two sets. That is, we want to introduce the entire set of cakes relative to each and every student. We will achieve this by means of a distributivity operator $\mathbf{dist}_{u}$ over students taking scope over a maximization operator $\mathbf{max}^{u'}$ operator over cakes. Note that the distributivity operator is anaphoric to the dref u introduced by a preceding maximization operator $\mathbf{max}^{u}$ over students, as shown in (25) below.

24. **a.** Every^{u} student ate from every$^{u'}$ cake. **b.** They$_{u}$ liked them$_{u'}$ (all) [9].
25. $\mathbf{max}^{u}([student\{u\}]);\ \mathbf{dist}_{u}(\mathbf{max}^{u'}([cake\{u'\}]));\ [eat\_from\{u, u'\}];\ [like\{u, u'\}]$

Intuitively, the update in (25) instructs us to perform the following operations on a given input matrix I:

- $\mathbf{max}^{u}([student\{u\}])$: add a new column u and store all the students in it;

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[8] Both maximization and distributivity are selective in the sense that they target a particular dref u over which they maximize or distribute), i.e. exactly in the sense in which the DPL/FCS/DRT-style dynamic generalized quantification introduced in chapters **2**, **4** and **5** is selective and, by being so, solves the proportion and weak / strong ambiguity problems which mar the notion of unselective quantification introduced in Lewis (1975).

[9] Another example with a similar 'Cartesian product' interpretation is *Every guest tasted every dish at the potluck party.*

- $\mathbf{dist}_u(\mathbf{max}^{u'}([cake\{u'\}]))$: look at each *u*-individual separately – more exactly, for each such individual *x*, look at that subpart of the matrix that has only *x* in column *u*; relative to each such sub-matrix, add a new column *u'* and store all the cakes in that column; then, take the union of all the resulting matrices: the big union matrix will associated every *u*-individual separately with each and every cake;
- $[eat\_from\{u, u'\}]; [like\{u, u'\}]$: test that, for each row in the big union matrix, the *u*-individual stored in that row ate from the *u'*-individual stored in that row; finally, test that, for every rowin the big union matrix, the *u*-individual stored in that row liked the *u'*-individual stored in that row.

A different kind of example indicating that we need selective distributivity operators over and above the unselective distributivity built into the atomic conditions[10] to obtain structure maximization is provided by the donkey sentence in (26) below. Intuitively, the donkey indefinite receives a strong reading, i.e. every farmer kicked *every* donkey he saw (and not only some). In particular, if two farmers happened to see the same donkeys, each one of them kicked each one the donkeys, i.e. we need to consider each farmer in turn and introduce every seen donkey with respect to each one of them. Again, this can be achieved by means of a $\mathbf{dist}_u$ operator over farmers taking scope over a $\mathbf{max}^{u'}$ operator over donkeys, as shown in (27) below.

26. Every^u farmer who saw a$^{u'}$ donkey kicked it$_{u'}$.
27. $\mathbf{max}^u([farmer\{u\}]; \mathbf{dist}_u(\mathbf{max}^{u'}([donkey\{u'\}, see\{u, u'\}]))); [kick\{u, u'\}]$

Notice that the example in (24) above indicates that we need a $\mathbf{dist}_u$ operator over the nuclear scope of *every student* (since we need to introduce every cake relative to each student), while the example (26) above indicates that we need a $\mathbf{dist}_u$ operator over the restrictor of *every farmer* (since we need to introduce every donkey that was seen relative to each farmer). We therefore expect our final definition of dynamic generalized determiners to contain two distributivity operators – and this is exactly how it will be.

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[10] Atomic conditions are unselectively distributive because they contain the universal quantifications over 'assignments' of the form $\forall i_s \in I(\dots)$, i.e. they unselectively target 'assignments' (i.e. cases in the sense of Lewis 1975) and not individuals or individual dref's, as the selectively distributive operator $\mathbf{dist}_u$ does.

The $\mathbf{max}_u$ and $\mathbf{dist}_u$ operators are defined in (28) and (31) below. Consider the definition of $\mathbf{max}^u$ first: the first conjunct in (28) introduces *u* as a new dref (i.e. [*u*]) and makes sure that each individual in *uJ* 'satisfies' *D*, i.e. we store *only* individuals that 'satisfy' *D*. The second conjunct enforces the maximality requirement: any other set *uK* obtained by a similar procedure (i.e. any other set of individuals that 'satisfies' *D*) is included in *uJ*, i.e. we store *all* the individuals that satisfy *D*.

28. $\mathbf{max}^u(D) := \lambda I_{st}.\lambda J_{st}.\ ([u];D)IJ \wedge \forall K_{st}(([u];D)IK \rightarrow uK_{u\neq\#}\subseteq uJ_{u\neq\#})$

29. $\mathbf{max}^{u'\sqsubseteq u}(D) := \mathbf{max}^{u'}([u'\sqsubseteq u];D)$

30. $I_{u=x} := \{i_s\in I\text{: } ui=x\}$

31. $\mathbf{dist}_u(D) := \lambda I_{st}.\lambda J_{st}.\ \forall x_e(I_{u=x}\neq\emptyset \leftrightarrow J_{u=x}\neq\emptyset) \wedge \forall x_e(I_{u=x}\neq\emptyset \rightarrow DI_{u=x}J_{u=x})$,

i.e. $\mathbf{dist}_u(D) := \lambda I_{st}.\lambda J_{st}.\ uI=uJ \wedge \forall x_e\in uI(DI_{u=x}J_{u=x})$

The basic idea behind *distributively* updating an input info state *I* with a DRS *D* is that we first partition the info state *I* and then *separately* update each partition cell (i.e. subset of *I*) with *D*.

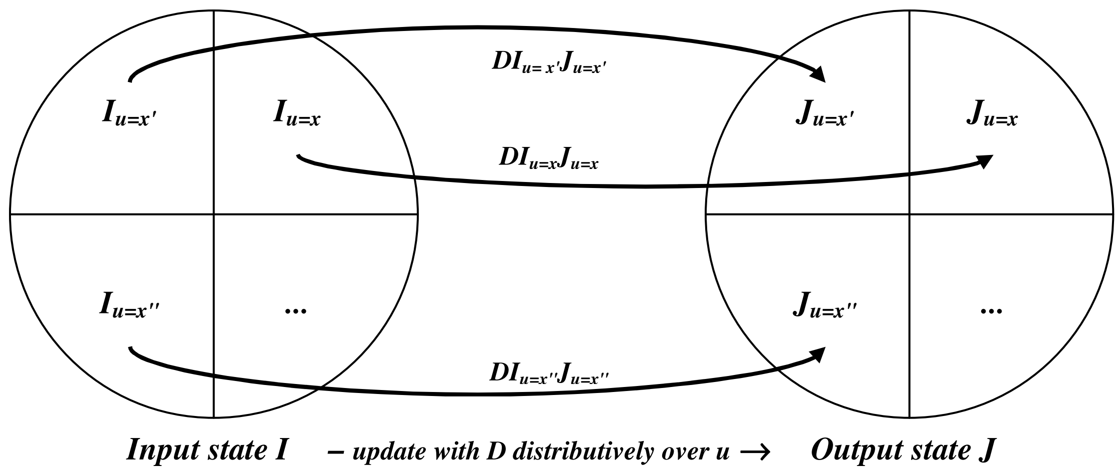
Moreover, the partition of the info state *I* is induced by the dref *u* as follows: consider the set of individuals $uI := \{ui\text{: } i\in I\}$; each individual *x* in the set *uI* generates one cell in the partition of *I*, namely the subset $\{i\in I\text{: } ui=x\}$. Clearly, the family of sets $\{\{i\in I\text{: } ui=x\}\text{: } x\in uI\}$ is a partition of the info state *I*.

Thus, updating an info state *I* with a DRS *D* *distributively* over a dref *u* means updating each cell in the *u*-partition of *I* with the DRS *D* and then taking the union of the resulting output info states. The first conjunct in definition (31) above, i.e. $uI=uJ$, is required to ensure that there is a bijection between the partition cells induced by the dref *u* over the input state *I* and the partition cells induced by *u* over the output state *J*; without this requirement, we could introduce arbitrary new values for *p* in the output state *J*, i.e. arbitrary new partition cells[11].

[11] Nouwen (2003): 87 was the first to observe that the first conjunct in this definition, namely $uI=uJ$, is necessary (the original definition in van den Berg 1996a: 145, (18) lacks it).

The second conjunct, i.e. $\forall x \in uI(DI_{u=x}J_{u=x})$, is the one that actually defines the distributive update: every partition cell in the input info state *I* is related by the DRS *D* to the corresponding partition cell in the output state *J*. The figure in (32) below schematically represents how the input state *I* is *u*-distributively updated with the DRS *D*[12].

32. **Updating info state *I* with *D* distributively over *u*.**



The definitions of generalized determiners and weak / strong indefinites are provided in (36), (37) and (38) below. For the justification of the account of weak / strong donkey ambiguities in terms of weak / strong indefinite articles, see chapter **5**.

33. $_{u}(D) := \lambda I_{st}.\lambda J_{st}.\ I_{u=\#}=J_{u=\#} \wedge I_{u\neq\#}\neq\emptyset \wedge \mathbf{dist}_u(D)I_{u\neq\#}J_{u\neq\#}$ [13]

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[12] Some properties of the distributivity operator (see also the appendix of this chapter):

(i) $\mathbf{dist}_u(D;\ D') = \mathbf{dist}_u(D);\ \mathbf{dist}_u(D')$, for any *D* and *D'* s.t. $\forall\langle I,J\rangle\in D(uI=uJ)$ and $\forall\langle I,J\rangle\in D'(uI=uJ)$ (i.e. $\mathbf{dist}_u$ distributes over dynamic conjunction)

(ii) $\mathbf{dist}_u(\mathbf{dist}_{u'}(D)) = \mathbf{dist}_{u'}(\mathbf{dist}_u(D))$

(iii) $\mathbf{dist}_u(\mathbf{dist}_u(D)) = \mathbf{dist}_u(D)$.

[13] Some properties of the $_{u}(\ldots)$ operator:

(i) $_{u}(D;\ D') = {}_{u}(D);\ {}_{u}(D')$, for any *D* and *D'* s.t. $\mathbf{dist}_u(D;\ D') = \mathbf{dist}_u(D);\ \mathbf{dist}_u(D')$

(ii) $_{u}(_{u}(D)) = {}_{u}(D)$

However, note that, in general, $_{u}(_{u'}(D)) \neq {}_{u'}(_{u}(D))$. Consider for example the info state *I* in (42) below: while it is true that $\langle I,I\rangle$ is in the denotation of $_{u}(_{u'}([u\Subset u]))$, it is not true that $\langle I,I\rangle$ is in the denotation of $_{u'}(_{u}([u\Subset u]))$. Moreover, we can easily construct an info state *I'* such that $\langle I',I'\rangle$ is in the denotation of $_{u'}(_{u}([u\Subset u]))$, but not in the denotation of $_{u}(_{u'}([u\Subset u]))$.

34. $_{\langle u\rangle}(D) := \lambda I_{st}.\lambda J_{st}.\ I_{u=\#}=J_{u=\#} \wedge (I_{u\neq\#}=\emptyset \rightarrow I=J) \wedge (I_{u\neq\#}\neq\emptyset \rightarrow \mathbf{dist}_u(D)I_{u\neq\#}J_{u\neq\#})$

35. $\mathbf{DET}\{u, u'\} := \lambda I_{st}.\ \mathbf{DET}(uI_{u\neq\#}, u'I_{u'\neq\#})$,

where **DET** is a static determiner.

36. $det^{u,u'\sqsubseteq u} \rightsquigarrow \lambda P_{\mathbf{et}}.\lambda P'_{\mathbf{et}}.\ \mathbf{max}^u(_{\langle u\rangle}(P(u)));\ \mathbf{max}^{u'\sqsubseteq u}(_{\langle u'\rangle}(P'(u')));\ [\mathbf{DET}\{u, u'\}]$

37. $a^{\mathrm{wk}:u} \rightsquigarrow \lambda P_{\mathbf{et}}.\lambda P'_{\mathbf{et}}.\ [u];\ _u(P(u));\ _u(P'(u))$

38. $a^{\mathrm{str}:u} \rightsquigarrow \lambda P_{\mathbf{et}}.\lambda P'_{\mathbf{et}}.\ \mathbf{max}^u(_u(P(u));\ _u(P'(u)))$

Note that the **max**-based definition of selective generalized quantification correctly predicts that anaphora to restrictor and nuclear scope sets is always anaphora to *maximal sets*, i.e. E-type anaphora (recall the Evans examples: *Few^u congressmen admire Kennedy and they$_u$ are very junior* and *Harry bought some^u sheep. Bill vaccinated them$_u$*[14]; see also (10), (11) and (12) above). The maximality of anaphora to quantifier sets follows automatically as a consequence of the fact that we need maximal sets to correctly compute the meaning of dynamic generalized quantifiers. This is one of the major results in van den Berg (1996a) and PCDRT preserves it[15].

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[14] See Evans (1980): 217, (7) and (8) (page references are to Evans 1985).

[15] That the restrictor set needs to be maximal is established by *every*-quantifications: to determiner the truth of *Every man left*, we need to have access to the set of all men. That the nuclear scope set also needs to be maximal, namely the maximal subset of the restrictor set that satisfies the nuclear scope update, is established by downward monotonic quantifiers (i.e. by determiners that are downward monotonic in their right argument); for example, *Few men left* intuitively means that, among the set of men, the *maximal* set of men that left is a *few*-subset, i.e. it is less than half of the set of men. In particular, if *Few men left* is true, then *Most men left* is false – and the use of maximal nuclear scope sets correctly predicts that.

If we were to use non-maximal subsets of the restrictor set of individuals, we would be able to capture the meaning of upward monotonic quantifiers, e.g. *Most* (*some*, *two*, *at least two*, etc.) *men left* can be interpreted as: introduce the maximal set of men (i.e. the maximal restrictor set); then, introduce some subset of the restrictor set that is a *most*-subset (i.e. it is more than half of the restrictor set) and that also satisfies the nuclear scope update. If you can do this, then the quantification update is successful. Note that, in this case, the nuclear scope set is not necessarily the maximal subset of the restrictor set that satisfies the nuclear scope update. The relevant definition is given in (i) below.

(i) $det^{u,u'\sqsubseteq u} \rightsquigarrow \lambda P_{\mathbf{et}}.\lambda P'_{\mathbf{et}}.\ \mathbf{max}^u(_{\langle u\rangle}(P(u)));\ [u' \mid u'\sqsubseteq u, \mathbf{DET}\{u, u'\}];\ _{\langle u'\rangle}(P'(u'))$

But this strategy will not work with downward monotonic quantifiers, e.g. *Few* (*no*, *at most two* etc.) *students left* cannot be interpreted as: introduce the maximal set of men (i.e. the maximal restrictor set); then, introduce some subset of the restrictor set that is a *few*-subset (i.e. it is less than half of the restrictor set, possibly empty) and that also satisfies the nuclear scope update (if the *few*-subset that was introduced is empty, we can assume that it vacuously satifies the nuclear scope update). We cannot do this because, even if we are successful in introducing a *few*-subset that satisfies the nuclear scope update, it can still be the

Moreover, this result is an important argument for a dynamic approach to generalized quantification in general and, in particular, for a dynamic approach to generalized quantification of the kind pursued in this chapter.

3.4. The Dummy Individual and Distributivity Operators

We have already established that the definition of generalized determiners in (36) above requires a distributivity operator $\mathbf{dist}_u$. The distributivity operator is contributed by the operators ${}_u(D)$ and ${}_{\langle u\rangle}(D)$ defined in (33) and (34) above. The question is: why do we need the additional conjuncts in the definition of these operators over and above distributivity?

To see the necessity of the first conjunct $I_{u=\#}=J_{u=\#}$ in (33) and (34), consider the simple sentence in (39) below, represented in (40) without the operator ${}_u(\ldots)$ and in (41) with the operator ${}_u(\ldots)$ [16].

39. A^u man fell in love with a$^{u'}$ woman.
40. $[u \mid man\{u\}]$; $[u' \mid woman\{u'\}, f\_i\_l\{u, u'\}]$
41. $[u]$; ${}_u([man\{u\}]$; $[u']$; ${}_{u'}([woman\{u'\}, f\_i\_l\{u, u'\}]))$

After processing sentence (39), we want our output info state to be such that each non-dummy u-man loves some non-dummy u'-woman and each non-dummy u'-woman loves some non-dummy u-man. However, if the conjunct $I_{u=\#}=J_{u=\#}$ is lacking – as it is lacking in (40) above –, we might introduce some u'-women relative to 'assignments' that

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case that a *most*-subset, for example, also satisfies the update, i.e. a successful update with *Few men left* does not rule out the possibility that *Most men left*, which is intuitively incorrect.

For the quantification *Few men left* to rule out the possibility that a *most*-subset of the restrictor also satisfies the nuclear scope update, we need to introduce the maximal nuclear scope set, i.e. the maximal subset of the restrictor that satisfies the nuclear scope update and only afterwards test that the two maximal sets are related by the static determiner. This is a direct consequence of the proposition relating witness sets and quantifier monotonicity in Barwise & Coopeer (1981): 104 (page references to Partee & Portner 2002).

In conclusion, to correctly computate the truth-conditions of generalized quantifications, the dynamic meaning of generalized determiners have to relate two maximal sets of individuals (i.e. the restrictor set and the nuclear scope set) – and this automatically and correctly predicts that E-type (i.e. unbound, 'quantifier external') anaphora to quantificational domains is maximal.

[16] These oversimplified representations are good enough for our current purposes. For the actual PCDRT analysis of this example, see (55) and (60) in section **3.6** below.

store the dummy individual # with respect to the dref u, see for example 'assignment' i_3 in (42) below.

42. Info State I	u (men)	u' (women)
i_1	x_1	y_1
i_2	x_2	y_2
i_3	#	y_3
i_4	#	#

Given that we ignore both i_3 and i_4 in the evaluation of the lexical relation *f_i_l*{u, u'}, y_3 can be any woman whatsoever (including a woman that is not loved by any man) – which can inadvertently falsify subsequent anaphoric sentences, e.g. the follow-up $She_{u'}$ *was pretty*, which might actually be true of y_1 and y_2, but not of y_3. The discourse $Every^u$ *man fell in love with* $a^{u'}$ *woman.* $They_{u'}$ *were pretty* provides a similar argument for the necessity of the first conjunct $I_{u=\#}=J_{u=\#}$ in (33) and (34).

The second conjunct $I_{u\neq\#}\neq\emptyset$ in the definition the operator $_u(\ldots)$ in (33) above encodes existential commitment. Note that the existential commitment associated with dref introduction is built into two distinct definitions: (**i**) the definition of lexical relations (see the conjunct $I_{u_1 \neq \#, \ldots, u_n \neq \#} \neq \emptyset$ in (22) above) and (**ii**) the definition of the operator $_u(\ldots)$ (see the conjunct $I_{u\neq\#}\neq\emptyset$ in (33)).

We need the former (i.e. the conjunct $I_{u_1 \neq \#, \ldots, u_n \neq \#}\neq\emptyset$ in the definition of lexical relations) because the pair <$\emptyset_{st}$, $\emptyset_{st}$> belongs to the denotation of [u] for any dref u (since both conjuncts in the definition of [u] are universal quantifications).

We need the latter (i.e. the conjunct $I_{u\neq\#}\neq\emptyset$ in the definition of the operator $_u(\ldots)$) because the definition of the $\mathbf{dist}_u$ operator is a universal quantification and is therefore trivially satisfied relative to the empty input info state $\emptyset_{st}$; that is, the pair <$\emptyset_{st}$, $\emptyset_{st}$> belongs to the denotation of $\mathbf{dist}_u(D)$ for any dref u and DRS D.

Thus, we capture the existential commitment associated with indefinites by using the operator $_u(\ldots)$ in their translation – see (37) and (38) above.

In contrast, there is no such existential commitment in the definition of the the operator $_{\langle u\rangle}(\ldots)$ in (34) above and, therefore, there is no such existential commitment in the definition of generalized determiners $det^{u,u'\sqsubseteq u}$ in (36). This enables us to capture the meaning of both upward and (especially) downward monotonic quantifiers by means of the same definition. The problem posed by downward monotonic quantifiers is that their nuclear scope set can or has to be empty.

For example, after a successful update with a $no^{u,u'\sqsubseteq u}$ quantification (e.g. *No man left*), the nuclear scope set is necessarily empty (recall that we use nuclear scope sets with built-in conservativity), i.e. the dref u' will always store only the dummy individual # relative to the output info state. This, in turn, entails that no lexical relation in the nuclear scope DRS that has u' as an argument can be satisfied (because the first conjunct of any such lexical relation is $I_{u_1 \neq \#, \ldots, u_n \neq \#} \neq \emptyset$ – see (22) above). Thus, we need the operator $_{\langle u\rangle}(\ldots)$ – more precisely, the second conjunct in its definition in (34) above – to resolve the conflict between the emptiness requirement enforced by a *no*-quantification and the non-emptiness requirement enforced by lexical relations.

Similarly, given that we use the same operator $_{\langle u\rangle}(\ldots)$ in the formation of restrictor sets, we predict that *John visited every Romanian colony* is true (although it might not always be felicitous) in case there are no Romanian colonies, i.e. in case the restrictor set of the *every*-quantification is empty.

Note that, despite the fact that definition (34) allows for empty restrictor and nuclear scope sets, we are still able to capture the fact that subsequent anaphora to such sets is infelicitous. This follows from: (**i**) the fact that lexical relations have a non-emptiness / existential requirement built in and (**ii**) pronouns will be defined by means of the operator $_{u}(\ldots)$ (see (44) below), which also has a non-emptiness / existential requirement built in.

Finally, note that the second conjunct the definition of $_{\langle u'\rangle}(\ldots)$ in (34) requires the identity of the input state I and the output state J. That is, the nuclear scope DRS of a successful $no^{u,u'\sqsubseteq u}$ quantification, i.e. $_{\langle u'\rangle}(P'(u'))$, will always be a test. Consequently, we correctly predict that anaphora to any indefinites in the nuclear scope of a $no^{u,u'\sqsubseteq u}$

quantification is infelicitous, e.g. *No*$^{u,u'\sqsubseteq u}$ *farmer owns a*$^{u''}$ *donkey. #It*$_{u''}$ *is unhappy / #They*$_{u''}$ *are unhappy* (or *Harry courts a girl*$^{u''}$ *at no*$^{u,u'\sqsubseteq u}$ *convention. #She*$_{u''}$ *is very pretty.*).

3.5. Singular Number Morphology on Pronouns

Let us turn now to the last component needed for the account of discourses (1) and (2), namely the representation of singular pronouns. Their PCDRT translation, provided in (44) below, has the expected Montagovian form: it is the distributive type-lift of the dref u, i.e. $\lambda P_{\mathbf{et}}.\ {}_u(P(u))$, with the addition of the condition $\mathbf{unique}\{u\}$, which is contributed by the singular number morphology and which requires uniqueness of the non-dummy value of the dref u relative to the current plural info state – see (43) below.

43. $\mathbf{unique}\{u\} := \lambda I_{st}.\ I_{u\neq\#}\neq\varnothing \wedge \forall i_s,i'_s\in I_{u\neq\#}(ui=ui')$

44. *she*$_u$ $\rightsquigarrow \lambda P_{\mathbf{et}}.\ [\mathbf{unique}\{u\}];\ {}_u(P(u))$

In contrast, plural pronouns do not require uniqueness, as shown in (45) below.

45. *they*$_u$ $\rightsquigarrow \lambda P_{\mathbf{et}}.\ {}_u(P(u))$

Singular and plural *anaphoric* definite descriptions – we need them to interpret the anaphoric DP *the girl* in (2c) above among others – are interpreted as shown in (46) and (47) below. They exhibit the same kind of unique/non-unique contrast as the pronouns.

46. *the_sg*$_u$ $\rightsquigarrow \lambda P_{\mathbf{et}}.\lambda P'_{\mathbf{et}}.\ [\mathbf{unique}\{u\}];\ {}_u(P(u));\ {}_u(P'(u))$

47. *the_pl*$_u$ $\rightsquigarrow \lambda P_{\mathbf{et}}.\lambda P'_{\mathbf{et}}.\ {}_u(P(u));\ {}_u(P'(u))$

The uniqueness enforced by the condition $\mathbf{unique}\{u\}$ is *weak* in the sense that it is relativized to the current plural info state. However, we can require *strong* uniqueness, i.e. uniqueness relative to the entire model, by combining the $\mathbf{max}^u$ operator and the condition $\mathbf{unique}\{u\}$ – as shown by the Russellian, non-anaphoric meaning for definite descriptions provided in (48) below, which, as expected from a Russellian analysis, requires both existence and strong uniqueness. This alternative meaning for definite articles is needed to interpret the non-anaphoric DP *the banquet* in (2b) above.

48. *the_sg*^{u} $\rightsquigarrow$ $\lambda P_{\mathbf{et}}.\lambda P'_{\mathbf{et}}.\ \mathbf{max}^u({}_u(P(u)));\ [\mathbf{unique}\{u\}];\ {}_u(P'(u))$ [17]

The PCDRT translation of proper names is provided in (49) below. The definitions of dynamic negation and truth are identical to the ones in chapter **5**, as shown by (50) and (51) respectively.

49. *Harvey*^{u} $\rightsquigarrow$ $\lambda P_{\mathbf{et}}.\ [u \mid u \Subset Harvey];\ {}_u(P(u))$,

where $Harvey := \lambda i_s.\ harvey_e$.

50. $\sim D := \lambda I_{st}.\ I \neq \emptyset \wedge \forall H_{st} \neq \emptyset (H \subseteq I \rightarrow \neg \exists K_{st}(DHK))$ [18]

51. A DRS D (of type $\mathbf{t} := (st)((st)t)$) is *true* with respect to an input info state I_{st} iff $\exists J_{st}(DIJ)$.

3.6. An example: Cross-Sentential Anaphora to Indefinites

I will conclude this section with the PCDRT analysis of the simple example in (39) above. The transitive verb *fall in love* is translated as shown in (52) below. Also, for simplicity, I will assume that both indefinites are weak and are therefore translated as

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[17] The plural counterpart of the Russellian singular definite article in (48) is provided in (i) below – the only difference is that we remove the **unique**$\{u\}$ condition from its singular counterpart, just as we did for plural pronouns and anaphoric plural definite articles in (45) and (47) above.

(i) *the_pl*^{u} $\rightsquigarrow$ $\lambda P_{\mathbf{et}}.\lambda P'_{\mathbf{et}}.\ \mathbf{max}^u({}_u(P(u)));\ {}_u(P'(u))$

Note that the Russellian plural definite translation in (i) above is identical to the simplified translation of every in (65) below (see section **4.1**), which preserves the intuitive equivalence between *every*-DP's and (distributive uses of) plural *the*-DP's, e.g. *Every student left* and *The students left*, already observed and captured in Link (1983).

[18] This definition of negation enables us to capture the interaction between negation and intra-sentential donkey anaphora in (i), (ii) and (iii) below (as already indicated in section **3.3** of chapter **5**) and also between negation and cross-sentential anaphora in (iv).

(i) Most farmers who own a^u donkey do not beat it$_u$.

(ii) Every farmer who owns a^u donkey doesn't feed it$_u$ properly.

(iii) Most house-elves who fall in love with a^u witch do not buy her$_u$ an$^{u'}$ alligator purse.

(iv) Every^u student bought several$^{u'}$ books. But they$_u$ didn't read (any of) them$_{u'}$.

shown in (53) below. The semantic composition[19] proceeds based on the syntactic structure schematically represented in (54) and yields the representation in (55).

52. fall_in_love ⇝ $\lambda Q'_{\mathbf{(et)t}}.\lambda v_{\mathbf{e}}.\ Q'(\lambda v'_{\mathbf{e}}.\ [f\_i\_l\{v, v'\}])$

53. *a*$^{wk:u}$ *man* ⇝ $\lambda P_{\mathbf{et}}.\ [u];\ _{u}([man\{u\}]);\ _{u}(P(u))$

a$^{wk:u'}$ *woman* ⇝ $\lambda P_{\mathbf{et}}.\ [u'];\ _{u'}([woman\{u'\}]);\ _{u'}(P(u'))$

54. *a*$^{wk:u}$ *man* [*fall_in_love* [*a*$^{wk:u'}$ *woman*]]

55. $[u];\ _{u}([man\{u\}]);\ _{u}([u'];\ _{u'}([woman\{u'\}]);\ _{u'}([f\_i\_l\{u, u'\}]))$

To simplify the representation in (55), I will introduce the abbreviations in (56) and (57) below. The reader can easily check that the identities in (58) and (59) hold.

56. $_{u}(C) := \lambda I_{st}.\ I_{u\neq\#}\neq\emptyset \wedge \forall x\in uI_{u\neq\#}(CI_{u=x})$,

where C is a condition (of type $(st)t$).

57. $_{u}(u_1, \ldots, u_n) := \lambda I_{st}.\lambda J_{st}.\ I_{u=\#}=J_{u=\#} \wedge I_{u\neq\#}[u_1, \ldots, u_n]J_{u\neq\#}$,

where $u\notin \{u_1, ..., u_n\}$ and $[u_1, ..., u_n] := [u_1]; ...; [u_n]$ [20].

58. $_{u}([C_1, \ldots, C_m]) = [_{u}(C_1), \ldots, {}_{u}(C_m)]$

59. $_{u}([u_1, ..., u_n \mid C_1, \ldots, C_m]) = [_{u}(u_1, ..., u_n) \mid {}_{u}(C_1), \ldots, {}_{u}(C_m)])$

Based on the identities in (58) and (59) and several fairly obvious simplifications, we obtain the final PCDRT translation of sentence (39), provided in (60) below. Based on the definition of truth in (51) above, we derive the truth-conditions in (61) below, which agree with our intuitions about the truth-conditions of sentence (39).

60. $[u, {}_{u}(u') \mid man\{u\}, {}_{u}(woman\{u'\}), {}_{u}(f\_i\_l\{u, u'\})]$

61. $\lambda I_{st}.\ I\neq\emptyset \wedge \exists x_e \exists y_e(man(x) \wedge woman(y) \wedge f\_i\_l(x, y))$

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[19] That is, the type-driven translation of example (39); for the precise definition, see section **5** of chapter **3**.

[20] That is: $[u_1, ..., u_n] := \lambda I_{st}.\lambda J_{st}.\ \exists H_1\ldots\exists H_{n-1}(I[u_1]H_1 \wedge \ldots \wedge H_{n-1}[u_n]J)$.

3.7. The Dummy Info State as Default Discourse Context

In general, I take the default context of interpretation for all discourses to be the singleton info state $\{i_\#\}$, where $i_\#$ is the 'assignment' that stores the dummy individual # relative to all individual dref's. When we apply the truth-conditions in (61) above to the default input info state $\{i_\#\}$, we obtain $\exists x_e \exists y_e(man(x) \wedge woman(y) \wedge f\_i\_l(x, y))$, i.e. precisely the classical first-order truth-conditions assigned to sentence (39).

Moreover, taking $\{i_\#\}$ to be the default context of interpretation enables us to capture the infelicity of discourse-initial anaphors, e.g. #*She*$_u$ *is pretty*, because multiple meaning components (in particular, the condition **unique**$\{u\}$, the lexical relation **pretty**$\{u\}$ and the operator $_u(\ldots)$) cannot be satisfied relative to the input info state $\{i_\#\}$.

Hence, the felicitous *deictic* use of a pronoun like *she*$_u$ requires us to *non-linguistically* update the default input info state $\{i_\#\}$ before processing the sentence containing the pronoun; intuitively, this update is contributed by the deixis associated with the pronoun (see Heim 1982/1988: 309 et seqq for a similar assumption[21]).

4. Quantificational Subordination in PCDRT

This section presents the PCDRT analysis of the contrast in interpretation between the discourses in (1) and (2) above.

4.1. Quantifier Scope

We start with the two possible quantifier scopings for the discourse-initial sentence (1a/2a). For simplicity, I will assume that the two scopings are due to the two different lexical entries for the ditransitive verb *court_at*, provided in (62) and (63) below. As chapter **5** showed, PCDRT is compatible with Quantifier Raising / Quantifying-In and, in

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[21] "If something has been mentioned before, there will always be a card for it in the file […] But does the file also reflect what is familiar by contextual salience? So far we have not assumed it does, but let us make the assumption now. […] An obvious implication is that files must be able to change, and in particular, must be able to have new cards added, without anything being uttered. For instance, if halfway through a conversation between A and B a dog comes running up to them and draws their attention, then that event presumably makes the file increase by a new card" (Heim 1982/1988: 309-310).

general, with any of the quantifier scoping mechanisms proposed in the literature, there is no need to use any of them for our current purposes.

Furthermore, I will assume that the syntactic structure of the sentence is the one schematically represented in (64) below.

62. *court_at*^{1} $\rightsquigarrow \lambda Q'_{\mathbf{(et)t}}.\lambda Q''_{\mathbf{(et)t}}.\lambda v_{\mathbf{e}}.\ Q'(\lambda v'_{\mathbf{e}}.\ Q''(\lambda v''_{\mathbf{e}}.\ [court\_at\{v, v', v''\}]))$

63. *court_at*^{2} $\rightsquigarrow \lambda Q'_{\mathbf{(et)t}}.\lambda Q''_{\mathbf{(et)t}}.\lambda v_{\mathbf{e}}.\ Q''(\lambda v''_{\mathbf{e}}.\ Q'(\lambda v'_{\mathbf{e}}.\ [court\_at\{v, v', v''\}]))$

64. *Harvey* [[*court_at*$^{1/2}$ [*a girl*]] [*every convention*]]

Thus, *court_at*^{1} assigns the indefinite *a girl* wide scope relative to *every convention*, while *court_at*^{2} assigns it narrow scope.

Turning to the meaning of the quantifier *every convention*, note that we can safely identify the restrictor dref u and the nuclear scope dref u' of any *every*$^{u,u'\sqsubseteq u}$-quantification: the definition in (36) above entails that, if J is an arbitrary output state of a successful *every*$^{u,u'\sqsubseteq u}$-quantification, u and u' have to be identical both with respect to value and with respect to structure, i.e. we will have that $\forall j_s \in J(uj=u'j)$. We can therefore conflate the two dref's and assume that *every* contributes only one, as shown in (65) below. I will also assume that the restrictor set of the *every*^{u_1}-quantification is non-empty, so I will replace the operator $_{\langle u \rangle}(\ldots)$ with the simpler operator $_u(\ldots)$.

65. *every*^{u_1} $\rightsquigarrow \lambda P_{\mathbf{et}}.\lambda P'_{\mathbf{et}}.\ \mathbf{max}^{u_1}(_{u_1}(P(u_1)));\ _{u_1}(P'(u_1))$

The PCDRT translations of the generalized quantifier *every*^{u_1} *convention* and of the indefinite *a*$^{\mathbf{wk}:u_2}$ *girl* (which, for the moment, I assume to be weak) are given in (66) and (67) below, followed by the compositionally derived representations of the two quantifier scopings of sentence (1a/2a), which are provided in (68) and (69).

To make the representations simpler, I will assume that the PCDRT translation of the proper name *Harvey* is $\lambda P_{\mathbf{et}}.P(Harvey)$ instead of the one provided in (49) above. The reader can easily convince herself that this simplification does not affect the PCDRT truth-conditions for the two discourses under consideration.

66. *every*^{u_1} *convention* ⇝ $\lambda P_{\mathbf{et}}.\ \mathbf{max}^{u_1}([convention\{u_1\}]);\ {}_{u_1}(P(u_1))$

67. *a*$^{\mathbf{wk}:u_2}$ *girl* ⇝ $\lambda P_{\mathbf{et}}.\ [u_2 \mid girl\{u_2\}];\ {}_{u_2}(P(u_2))$

68. (*a*$^{\mathbf{wk}:u_2}$ *girl>>every*^{u_1} *convention*)

$[u_2 \mid girl\{u_2\}];\ {}_{u_2}(\mathbf{max}^{u_1}([convention\{u_1\}]));\ [{}_{u_2}(court\_at\{Harvey, u_2, u_1\})]$

69. (*every*^{u_1} *convention>>a*$^{\mathbf{wk}:u_2}$ *girl*)

$\mathbf{max}^{u_1}([convention\{u_1\}]);\ [{}_{u_1}(u_2) \mid {}_{u_1}(girl\{u_2\}),\ {}_{u_1}(court\_at\{Harvey, u_2, u_1\})]$

The reader can check that the (truth-conditions derived by the) representations in (68) and (69) are the intuitively correct ones. I will examine them only in informal terms.

The "wide-scope indefinite" representation in (68) updates the default input info state $\{i_\#\}$ as follows. First, we introduce some non-empty (i.e. non-dummy) set of individuals relative to the dref u_2. Then, we test that each u_2-individual is a girl. Then, relative to each u_2-individual, we introduce the non-empty set containing all and only conventions and store it relative to the dref u_1. Finally, we test that, for each u_2-girl, for each of the corresponding u_1-conventions (which, in this case, means: for every convention), Harvey courted the girl currently under consideration at the convention currently under consideration.

By the time we are done processing (68), the output info state contains a non-empty set of u_2-girls that where courted by Harvey at every convention and, relative to each u_2-girl, u_1 stores the set of all conventions.

The "narrow-scope indefinite" representation in (69) updates the default input info state $\{i_\#\}$ as follows. First, we introduce the non-empty set of individuals containing all and only conventions relative to the dref u_1. Then, for each u_1-convention, we introduce a u_2-set of individuals. Finally we test that, for each u_1-convention, each of the corresponding u_2-individuals are girls and are such that Harvey courted them at the convention currently under consideration.

By the time we are done processing (69), the output info state stores the set of all conventions under the dref u_1 and, relative to each u_1-convention, the dref u_2 stores a non-empty set of girls (possibly different from convention to convention) that Harvey courted at that particular convention.

4.2. Quantifier scope and Singular Anaphora, Cross-Sententially

It is now easy to see how sentence (1b) – and, in particular, the singular number morphology on the pronoun *she*$_{u_2}$ – forces the "indefinite wide-scope" reading for the preceding sentence (1a): the condition $\mathbf{unique}\{u_2\}$ effectively *conflates* the two readings by requiring the set of u_2-girls obtained after processing (68) or (69) above to be a singleton. This requirement leaves untouched the truth-conditions derived on the basis of (68) – but makes the truth-conditions associated with (69) above strictly stronger.

The PCDRT translation of the pronoun and the compositionally derived representation of sentence (1b) are provided in (70) and (71) below. For convenience, I provide the two complete representations of discourse (1) in (72) and (73) below.

70. *she*$_{u_2}$ $\leadsto \lambda P_{\mathbf{et}}.\ [\mathbf{unique}\{u_2\}];\ {}_{u_2}(P(u_2))$

71. $[\mathbf{unique}\{u_2\},\ very\_pretty\{u_2\}]$

72. (*a*$^{\mathbf{wk}:u_2}$ *girl>>every*^{u_1} *convention*)

    $[u_2 \mid girl\{u_2\}];\ {}_{u_2}(\mathbf{max}^{u_1}([convention\{u_1\}]));$

    $[{}_{u_2}(court\_at\{Harvey, u_2, u_1\}),\ \mathbf{unique}\{u_2\},\ very\_pretty\{u_2\}]$

73. (*every*^{u_1} *convention>>a*$^{\mathbf{wk}:u_2}$ *girl*)

    $\mathbf{max}^{u_1}([convention\{u_1\}]);$

    $[{}_{u_1}(u_2) \mid {}_{u_1}(girl\{u_2\}),\ {}_{u_1}(court\_at\{Harvey, u_2, u_1\}),\ \mathbf{unique}\{u_2\},\ very\_pretty\{u_2\}]$

4.3. Quantifier Scope and Singular Anaphora, Intra-Sententially

In contrast, sentence (2b) contains the adverb of quantification *always*, which can take scope above or below the singular pronoun *she*; in the former case, the u_2-uniqueness requirement is weakened (and, basically, neutralized) by being relativized to u_1-conventions.

More precisely, I take the meaning of *always* to be universal quantification over an anaphorically retrieved restrictor, as shown in (74) below. Since *always* is basically interpreted as *every*, I provide a simplified translation that conflates the restrictor and nuclear scope dref's – much like the simplified translation for *every* in (65) above conflated them. The general format for the interpretation of quantifiers that anaphorically retrieve their restrictor set is provided in (75).

74. *always*$_{u_1}$ $\rightsquigarrow \lambda P_{\mathbf{et}}.\ {}_{u_1}(P(u_1))$

75. *det*$_u{}^{u' \sqsubseteq u}$ $\rightsquigarrow \lambda P_{\mathbf{et}}.\ \mathbf{max}^{u' \sqsubseteq u}({}_{\langle u' \rangle}(P(u')));\ [\mathbf{DET}\{u, u'\}]$

The restrictor dref of *always* in (2b) is the nuclear scope dref of the quantifier *every*^{u_1} *convention* in the preceding sentence (2a). To see that *always* is indeed anaphoric to the nuclear scope and not to the restrictor dref of *every*, we need to consider other determiners that do not effectively identity them, e.g. *most* in (76) below. In this case, it is intuitively clear that *always* quantifies over the conventions at which Harvey courts a girl (the nuclear scope dref) and not over all conventions (the restrictor dref).

76. **a.** Harvey courts a girl at most conventions.
   b. She always comes to the banquet with him.

The definite description *the banquet* in (2b) is intuitively a Russellian definite description (see (48) above), which contributes existence and a relativized (i.e. anaphoric) form of uniqueness: we are talking about a *unique* banquet *per convention*. The relevant meaning for the definite article is given in (77) below.

77. *the_sg*$_{u_1}{}^{u_3}$ $\rightsquigarrow \lambda P_{\mathbf{et}}.\lambda P'_{\mathbf{et}}.\ {}_{u_1}(\mathbf{max}^{u_3}({}_{u_3}(P(u_3)));\ [\mathbf{unique}\{u_3\}];\ {}_{u_3}(P'(u_3)))$

The relativized uniqueness is captured by the fact that the $\mathbf{unique}\{u_3\}$ condition is within the scope of the ${}_{u_1}(\ldots)$ operator [22]. Thus, *the banquet* is in fact interpreted as a

[22] Incidentally, note that the definite article *the_sg*$_{u_1}{}^{u_3}$ is anaphoric to the restrictor set u_1 of the *every*-quantification in the preceding sentence (2a) – unlike *always*, which is anaphoric to the nuclear scope set. To see this, we have to consider determiners like *most* that do not conflate their restrictor and nuclear scope

possessive definite description of the form *its* $_{u_1}{}^{u_3}$ *banquet*, or, more explicitly, of the form *the* $_{u_1}{}^{u_3}$ *banquet of it* $_{u_1}$, where *it* $_{u_1}$ is anaphoric to u_1-conventions. The PCDRT translation of the definite description, obtained based on the translations in (77) and (78), is provided in (79) below.

78. *banquet of it* $_{u_1}$ $\rightsquigarrow \lambda v_{\mathbf{e}}.\ [banquet\{v\}, of\{v, u_1\}]$

79. *the_sg* $_{u_1}{}^{u_3}$ *banquet of it* $_{u_1}$ $\rightsquigarrow$

$$\lambda P_{\mathbf{et}}.\ {}_{u_1}(\mathbf{max}^{u_3}([banquet\{u_3\}, of\{u_3, u_1\}]); [\mathbf{unique}\{u_3\}];\ {}_{u_3}(P(u_3)))$$

However, to exhibit the interaction between the adverb *always* $_{u_1}$ and the pronoun *she* $_{u_2}$ in a simpler and more transparent way, I will assume that sentence (2b) contributes a dyadic relation of the form *come_with_Harvey_to_the_banquet_of* that relates girls and conventions. Just like *court_at*, this dyadic relation can be translated in two different ways, corresponding to the two possible relative scopes of *she* $_{u_2}$ and *always* $_{u_1}$ (that is, I employ the same scoping technique as the one used for sentence (1a/2a) in (62) and (63) above). The two different translations are provided in (80) and (81) below. The basic syntactic structure of sentence (2b) is provided in (82).

80. *come_to_banquet_of*^{1} $\rightsquigarrow \lambda Q_{\mathbf{(et)t}}.\lambda Q'_{\mathbf{(et)t}}.$

$$Q'(\lambda v'_{\mathbf{e}}.\ Q(\lambda v_{\mathbf{e}}.\ [come\_to\_banquet\_of\{v', v\}]))$$

81. *come_to_banquet_of*^{2} $\rightsquigarrow \lambda Q_{\mathbf{(et)t}}.\lambda Q'_{\mathbf{(et)t}}.$

$$Q(\lambda v_{\mathbf{e}}.\ Q'(\lambda v'_{\mathbf{e}}.\ [come\_to\_banquet\_of\{v', v\}]))$$

82. *she* [[*always*] *come_to_banquet_of*$^{1/2}$]

The first lexical entry *come_to_banquet_of*^{1} gives the pronoun *she* $_{u_2}$ wide scope over the adverb *always* $_{u_1}$, while the second lexical entry *come_to_banquet_of*^{2} gives the pronoun narrow scope relative to the adverb. The corresponding, compositionally derived PCDRT representations are provided in (83) and (84) below.

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dref's. So, consider discourse (76) again: intuitively, there is a unique banquet at *every* convention, not only at the majority of conventions where Harvey courts a girl.

83. (*she* $_{u_2}$ >>*always* $_{u_1}$)

[**unique**$\{u_2\}$, $_{u_2}$(*come_to_banquet_of*$\{u_2, u_1\}$)]

84. (*always* $_{u_1}$ >>*she* $_{u_2}$)

[$_{u_1}$(**unique**$\{u_2\}$), $_{u_1}$(*come_to_banquet_of*$\{u_2, u_1\}$)]

Thus, there are two possible representations for sentence (2a) – see (68) and (69) above – and two possible representations for sentence (2b) – given in (83) and (84) above. Hence, there are four possible representations for discourse (2) as a whole.

Out of the four possible combinations, three boil down to effectively requiring the indefinite *a*$^{\mathbf{wk}:u_2}$ *girl* to take wide scope over the quantifier *every*^{u_1} *convention*. This can happen if: (**i**) we assign the representation in (68) to sentence (2a), in which case it does not matter which of the two representations in (83) and (84) we assign to sentence (2b), or (**ii**) we assign the representation in (83) to sentence (2b), which, as we have already shown for discourse (1) (see section **4.2** above), effectively identifies the two possible representations of sentence (2a).

We are left with the fourth combination (69) + (84), i.e. *every*^{u_1} *convention>>a*$^{\mathbf{wk}:u_2}$ *girl* + *always* $_{u_1}$ >>*she* $_{u_2}$, which is given in (85) below and which provides the desired "narrow-scope indefinite" reading that is available for discourse (2), but not for (1).

85. $\mathbf{max}^{u_1}$([*convention*$\{u_1\}$]); [$_{u_1}$(u_2)]; [$_{u_1}$(*girl*$\{u_2\}$), $_{u_1}$(*court_at*$\{$*Harvey*$, u_2, u_1\}$)];

[$_{u_1}$(**unique**$\{u_2\}$), $_{u_1}$(*come_to_banquet_of*$\{u_2, u_1\}$)]

Intuitively, the PCDRT representation in (85) instructs us to modify the input info state $\{i_\#\}$ by introducing the set of all conventions relative to the dref u_1, followed by the introduction of a non-empty set of u_2-individuals relative to each u_1-convention. The remainder of the representation tests that, for each u_1-convention, the corresponding u_2-set is a singleton set consisting of a girl that is courted by Harvey at the u_1-convention currently under consideration and that comes with him at the banquet of said u_1-convention.

5. Summary

PCDRT enables us to formulate in classical type logic a compositional dynamic account of the intra- and cross-sentential interaction between generalized quantifiers, anaphora and number morphology exhibited by the quantificational subordination discourses in (1) and (2) above from Karttunen (1976).

The main proposal is that plural info states together with a suitable dynamic reformulation of independently motivated denotations for generalized determiners and number morphology in static Montague semantics enables us to account for quantificational subordination in terms of anaphora to quantifier domains and, consequently, for the contrast in interpretation between the discourses in (1) and (2) above.

The cross-sentential interaction between quantifier scope and anaphora, in particular the fact that a *singular* pronoun in the second sentence can disambiguate between the two readings of the first sentence, can be captured by plural information states because they enable us to store both quantifier domains (i.e. values) and quantificational dependencies (i.e. structure), pass them across sentential boundaries and further elaborate on them, e.g. by letting a pronoun constrain the cardinality of a previously introduced quantifier domain.

In the process, we were also able to show how the definite descriptions in sentences (2b) and (2c) can be analyzed and also how natural language quantifiers enter structured anaphoric connections as a matter of course, usually functioning simultaneously as both indefinites and pronouns.

6. Comparison with Alternative Approaches

6.1. Cross-Sentential Anaphora and Uniqueness

In this section (and the following one), I will briefly indicate some of the ways in which PCDRT relates to the previous literature on uniqueness effects associated with singular anaphora (see Evans 1977, 1980, Parsons 1978, Cooper 1979, Heim 1982/1988, Kadmon 1987, 1990, Neale 1990 and Roberts 2003 among others).

As indicated in section **3.5** of the present chapter (see also section **3.4** of chapter **5**), the uniqueness enforced by the condition **unique**$\{u\}$ is weak in the sense that it is relativized to the current plural info state. However, we can require *strong* uniqueness, i.e. uniqueness relative to the entire model, by combining the $\mathbf{max}^{u}$ operator and the condition **unique**$\{u\}$ – as, for example, in the PCDRT translation for Russellian, non-anaphoric definite descriptions provided in (48) above.

The same $\mathbf{max}^{u}$ + **unique**$\{u\}$ strategy can be employed to capture the strong uniqueness intuitions associated with the "narrow-scope indefinite" reading of the quantificational subordination discourse in (2) above, i.e. the fact that discourse (2) as a whole implies that Harvey courts a *unique* girl *per convention*.

In more detail: we have assumed throughout this chapter (for simplicity) that the indefinite *a girl* in (2a) receives a weak reading – but, if we assume that the indefinite has a strong / maximal reading (see the translation in (38) above), we can capture the above mentioned uniqueness intuitions. The PCDRT representation of the "narrow-scope strong indefinite" reading is provided in (86) below, which differs from the representation in (85) above only with respect to the presence of the additional maximization operator $\mathbf{max}^{u_2}$ contributed by the strong indefinite.

86. $\mathbf{max}^{u_1}([convention\{u_1\}]);\ {}_{u_1}(\mathbf{max}^{u_2}([girl\{u_2\}, court\_at\{Harvey, u_2, u_1\}]));$
$[{}_{u_1}(\mathbf{unique}\{u_2\}),\ {}_{u_1}(come\_to\_banquet\_of\{u_2, u_1\})]$

The strong uniqueness effect emerges as a consequence of the *combined* meanings assigned to the strong indefinite and the singular pronoun: the strong indefinite makes sure (by $\mathbf{max}^{u_2}$) that, with respect to each u_1-convention, the dref u_2 stores all the girls courted by Harvey at that convention; the singular pronoun subsequently requires (by **unique**$\{u_2\}$) that the set of u_2-individuals stored relative to each u_1-individual is a singleton set. Together, the strong indefinite and the singular pronoun require that, at each u_1-convention, Harvey courts exactly one girl, which the dref u_2 stores.

Thus, PCDRT can capture the intuition that discourse (2) is interpreted as talking about conventions at which Harvey courts a unique girl (possibly different from convention to convention). Moreover, the fact that, in PCDRT, the uniqueness implications are a consequence of *combining* the meanings of the indefinite and of the singular anaphor captures the observation in Kadmon (1990): 279-280 that "[…] indefinite NP's don't always have unique referents. […] When anaphora is attempted, however, the uniqueness effect always shows up".

In a sense, this observation is literally captured in PCDRT: singular pronouns always contribute a **unique**$\{u\}$ condition. However, whether this condition yields strong uniqueness depends on the weak / strong reading of the antecedent indefinite. Against Kadmon, I take this variation to be a welcome prediction since it converges with the wavering uniqueness intuitions that native speakers have with respect to various cases of singular cross-sentential anaphora (I will return to this issue presently).

The very same ingredients employed in PCDRT to derive the (relativized) uniqueness effects in quantificational subordination also provide an account of the (absolute / non-relativized) uniqueness intuitions associated with the well-known example in (87) below.

87. There is a$^{\mathbf{str}:u}$ doctor in London and he$_u$ is Welsh.
    (Evans 1980: 222, (26)[23])
88. $\mathbf{max}^u([doctor\{u\}, in\_London\{u\}]); [\mathbf{unique}\{u\}, Welsh\{u\}]$

In contrast, the weak and strong readings for the indefinite article in example (89) below (from Heim 1982/1988: 28, (14a)) are truth-conditionally indistinguishable in PCDRT[24], i.e. there are no strong uniqueness implications – and correctly so. Thus, PCDRT can also account for the difference between the interpretations of (87) and (89).

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[23] Page references are to Evans (1985).

[24] The weak / strong contrast associated with an indefinite has truth-conditional effects only if there is anaphora to that indefinite.

89. There is a doctor$^{\mathbf{wk/str}:u}$ who is Welsh in London. [25]

Finally, given that indefinite articles are associated with both a weak and a strong meaning enables us to account for the observation in Heim (1982): 31 that singular cross-sentential anaphora is not necessarily associated with uniqueness implications, as shown by the narration-type example in (90) below (from Heim (1982): 31, (29)).

90. There was a$^{\mathbf{wk}:u}$ doctor in London. He$_u$ was Welsh…

Summarizing, the hypothesis that the indefinite article is ambiguous between a weak and a strong reading together with proposal that singular number morphology on pronouns contributes a **unique** condition enables PCDRT to capture the three-way contrast between (87), (89) and (90) above. In particular, the contrast between (87) and (90) is due to what reading is associated with the indefinite in each particular case. PCDRT does not have anything to say about this choice – and, I think, rightfully so: as much of the literature observes (Heim 1982/1988, Kadmon 1990, Roberts 2003 among others), the choice is sensitive to various factors that are pragmatic in nature and / or have related to the global structure of the discourse (e.g. that (90) is a narrative, while (87) is not).

Thus, unlike Heim (1982) and classical DRT / FCS / DPL in general, PCDRT can capture the uniqueness intuitions (sometimes) associated with cross-sentential singular anaphora – and the ingredients of the analysis, in particular the two meanings associated with the indefinite article, are independently motivated by mixed reading donkey sentences (see chapter **5** above).

Moreover, the overall account is compositional and the **unique**$\{u\}$ condition contributed by singular number morphology on anaphors is a local constraint of the same kind as ordinary lexical relations, in contrast to the non-local and non-compositional[26]

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[25] PCDRT also makes correct predictions with respect to the similar examples in (i) and (ii) below, due to Heim (1982): 28, (27) and (27a).

(i) A wine glass broke last night. It had been very expensive.

(ii) A wine glass which had been very expensive broke last night.

[26] At least, not compositional in any obvious way.

uniqueness condition proposed in Kadmon (1990) to account for such uniqueness effects[27].

Also, unlike Kadmon (1990) (see the contrast between the preliminary and the final version of the uniqueness condition in Kadmon 1990[28]), PCDRT captures the constrast between the *absolute* and *relativized* uniqueness effects instantiated by (87) (where the doctor is absolutely unique) and (2) above (where there is a unique girl per convention) without any additional stipulations.

In particular, relativized uniqueness is a consequence of the distributivity operators contributed by the quantifier taking scope over the singular pronoun – and these distributivity operators are independently motivated by the scopal interaction between multiple quantifiers and by the interaction between generalized quantification and donkey anaphora (see the discussion in section **3.3** above).

Finally, the fact that indefinite articles are analyzed in PCDRT as being associated with both a weak and a strong meaning (independently motivated by mixed reading donkey sentences) adds the needed flexibility to account for the observation that cross-sentential anaphora is not always associated with uniqueness implications, as shown by the contrast between (87) and (90) above.

6.2. Donkey Anaphora and Uniqueness

The uniqueness implications associated with intra-sentential singular donkey anaphora are, by and large, just as unstable as the ones associated with cross-sentential singular anaphora.

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[27] This is the preliminary (simpler) version of the uniqueness condition in Kadmon (1990): 284, (30): "A definite NP associated with a variable X in DRS K is used felicitously only if for every model M, for all embedding functions f, g verifying K relative to M, $f(X)=g(X)$".

[28] The preliminary version of the uniqueness condition is provided in fn. **27** above. The final version of the uniqueness condition is as follows: "Let α be a definite NP associated with a variable Y, let K_{loc} be the local DRS of α, and let K be the highest DRS s.t. K is accessible from K_{loc} and $Y \in U_K$. α is used felicitously only if for every model M, for all embedding functions f, g verifying K relative to M, if $\forall X \in B_K\, f(X)=g(X)$ then $f(Y)=g(Y)$" (Kadmon 1990: 293, (31)), where $B_K := \{X: \exists K'$ accessible from K s.t. $K' \neq K$ and $X \in U_{K'}\}$.

On the one hand, the examples in (91) and (92) below exhibit uniqueness effects – more precisely: uniqueness effects relativized to each particular value of 'main' generalized determiner of each sentence (i.e. *most*, *every* and *every* respectively).

91. Every man who has a^u son wills him_u all his money.
    (Parsons 1978: 19, (4), attributed to B. Partee)
92. Every man who has a^u daughter thinks she_u is the most beautiful girl in the world.
    (Cooper 1979: 81, (60))

On the other hand, the examples in (93), (94), (95) and (96) below do not seem to exhibit uniqueness effects[29]. Note in particular that there are no uniqueness effects associated even with the *weak* donkey anaphora *$a^{u'}$ credit card-$it_{u'}$* in (96) (for more discussion of this observation, see chapter **5** above).

93. Every farmer who owns a^u donkey beats it_u.
94. Most people that owned a^u slave also owned his_u offspring.
    (Heim 1990: 162, (49))
95. No parent with a^u son still in high school has ever lent him_u the car on a weeknight.
    (Rooth 1987: 256, (48))
96. Every person who buys a^u TV and has $a^{u'}$ credit card uses $it_{u'}$ to pay for it_u.

In general, previous accounts of donkey anaphora are designed to account either for the first set of examples, which exhibit uniqueness (e.g. Parsons 1978, Cooper 1979, Kadmon 1990 among others), or for the second set of examples, which do not (e.g. Kamp 1981, Heim 1982/1988, 1990, Neale 1990, Kamp & Reyle 1993 among others). This is not to say that these approaches cannot be amended to account for a broader range of data – the point is only that the basic architecture of the theory is such that either uniqueness or non-uniqueness follows from it.

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[29] Kadmon (1990): 307 takes examples like (93) and (94) above to exhibit uniqueness – see also example (48) in Kadmon 1990: 307, repeated in (i) below. At the same time, Kadmon (1990): 308-309 mentions that some informants disagree and "treat [(i)] as if it said 'at least one dog'; for them, [(i)] doesn't display a uniqueness effect".

(i) Most women who own a dog talk to it.

In this section, I argue that the PCDRT combination of plural information states (plus maximization) on the one hand and the **unique** condition (plus distributivity) on the other hand makes for a flexible theory that can accommodate both kinds of donkey examples in a natural way. The main idea will be that all these resources enable us to 'partition' the restrictor of a generalized quantification in various ways and, depending on this 'partitioning', the morphologically singular anaphors in the nuclear scope of the generalized quantification contribute uniqueness or not.

The intuition that the uniqueness effects associated with donkey anaphora are dependent on how we 'think' about the restrictor of the generalized quantification is by no means new – it underlies the notion of *cases* in Lewis (1975), the use of *minimal situations* in Heim (1990) (among others) and the quantification over *instances* in Kadmon (1990) (see Kadmon 1990: 301). Thus, in this section, I argue that PCDRT enables us to formulate in a new and intuitive formalization of this familiar intuition.

Singular Donkey Anaphora Does Not Always Imply Uniqueness

The assumption that singular donkey anaphora can involve *non-singleton* sets has been repeatedly challenged because singular donkey anaphora seems to be intuitively associated with a kind of uniqueness implication[30]. Relative-clause donkeys in particular (like (1) and (2) above) are claimed to be associated with uniqueness presuppositions: some authors (e.g. Kanazawa 2001: 391, fn. 5) actually distinguish between relative-clause and conditional donkey sentences and claim that the former but not the latter contribute some form of uniqueness.

However, this is not the whole story. First, the uniqueness intuitions associated with relative-clause donkeys are much weaker (if at all present) when we consider examples with *multiple* donkey indefinites like (96) above, i.e., in a sense, relative-clause donkey sentences that are closer in form to conditional donkey sentences.

Second, even the proponents of uniqueness have to concede that donkey uniqueness is of a rather peculiar kind. One of the main debates revolves around the 'sage plant'

[30] For recent discussion, see Kanazawa (2001) and Geurts (2002).

example in (97) below which, on the face of it, strongly argues against donkey uniqueness.

97. Everybody who bought a^u sage plant here bought eight others along with it$_u$. (Heim 1982/1988: 89, (12))

Kadmon (1990): 317 conjectures that the donkey anaphora in (97) still contributes a uniqueness presupposition, but the "speakers accept this example because it can't make any difference to truth conditions which sage plant the pronoun *it* stands for, out of all the sage plants that a buyer *x* bought (for each buyer *x*)".

But, as Heim (1990): 161 points out, Kadmon's 'supervaluation'[31] analysis makes incorrect predictions with respect to the example in (95) above from Rooth 1987: intuitively, sentence (95) is falsified by any parent who has a son in high school and who has lent him the car on a weeknight even if said parent has another son who never got the car – which is to say that it *does* make a difference in this case which son the pronoun *him*$_u$ in (95) stands for[32].

This being said, example (91) above does seem to exhibit uniqueness implications – so, an empirically adequate account of donkey anaphora should be flexible enough to accommodate the wavering nature of the uniqueness intuitions associated with it.

Capturing the Wavering Nature of the Uniqueness Intuitions

As it now stands, the revised version of PCDRT introduced in this chapter predicts that donkey anaphora is associated with relativized uniqueness implications, i.e. it can account for the uniqueness intuitions associated with (91) above. As shown in (98) below, relativized uniqueness emerges as a consequence of the interaction between: (**i**) the distributivity operators contributed by selective generalized determiners, (**ii**) the maximization contributed by the strong reading of the indefinite and (**iii**) the **unique** condition contributed by the singular pronoun.

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[31] The connection with supervaluation treatments of vagueness is due to Mats Rooth – see Heim (1990): 160, fn. 11.

[32] For more discussion, see also Geurts (2002): 145 et seqq.

98. Every$^{u'}$ man who has a$^{\mathbf{str:}u}$ son wills him$_u$ all his money.

$\mathbf{max}^{u'}({}_{u'}([man\{u'\}];\ \mathbf{max}^{u}([son\{u\},\ have\{u',\ u\}])));$

${}_{u'}([\mathbf{unique}\{u\},\ will\_all\_money\{u',\ u\}])$[33]

Parsons (1978) considers the uniqueness effects associated with the donkey sentence in (91) above and suggests two different ways to capture them. The above PCDRT analysis can be seen as an implementation of the first suggestion:

> "One might suggest that the feeling of inappropriateness [of sentence (91) when taken to be talking about men that have more than one son] comes explicitly from the use of the pronoun. How would that work? Well, one purported meaning of 'a' is 'one', in the sense of 'exactly one'. Usually this is thought to be a presupposition, implication, or implicature of the utterance rather than part of the content of what is said. But perhaps the use of a singular pronoun can make the import part of the official content.
> The suggestion then is that 'a' can mean either 'at least one' or 'exactly one'. Normally it means the former, but certain grammatical constructions force the latter reading. The former reading is the 'indefinite' one, and the latter is the 'definite' one."
> (Parsons 1978: 19)

Interestingly, Parson's second suggestion is the one that is taken up by D-/E-type approaches that take pronouns to be numberless Russellian definite descriptions (e.g. Neale 1990)[34].

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[33] An unfortunate consequence of the fact that the **unique**$\{u\}$ condition contributed by the pronoun is taken to be part of the assertion is that the PCDRT representation in (98) is true only if every man has exactly one son, while, intuitively, the quantification should be restricted to men that have only one son. That is, the intuitively correct representation for (91) is the one in (i) below, where the **unique**$\{u\}$ condition occurs in the restrictor. This representation can be obtained if we assume that the **unique**$\{u\}$ condition is presupposed and that presuppositions triggered in the nuclear scope of tripartite quantificational structures can be accommodated in the restrictor (both assumptions, i.e. that number morphology on pronouns is presuppositional and that nuclear scope presuppositions can be accommodated in the restrictor, are independently assumed and motivated in the literature – see for example Beaver & Zeevat 2006, Heim 2005 and references therein).

(i) $\mathbf{max}^{u'}({}_{u'}([man\{u'\}];\ \mathbf{max}^{u}([son\{u\},\ have\{u',\ u\}]);\ [\mathbf{unique}\{u\}]));\ {}_{u'}([will\_all\_money\{u',\ u\}]).$

[34] "Sometimes 'the' doesn't mean 'exactly one', but rather 'at least one' or 'every'. It means 'at least one' in *everyone must pay the clerk five dollars* and it means 'every' in *you should always watch out for the other driver*. Or something like this. So perhaps the treatment of pronouns as paraphrases is correct, but we have to tailor the meaning of 'the' for the situation at hand. For example, in our sample sentence we need to read *the donkey he owns* as *every donkey he owns*. This response would involve specifying some method for determining which reading of *the* is appropriate in a given paraphrase; I haven't carried this out" (Parsons 1978: 20).

Thus, the version of PCDRT proposed in this chapter (chapter **6**) sides with the "uniqueness" approaches (e.g. Parsons 1978, Cooper 1979, Kadmon 1990 among others) – and therefore accounts for only one of the two sets of data. In contrast, the version of PCDRT proposed in the previous chapter (chapter **5**), which does not take singular pronouns to contribute a **unique** condition, sides with the "non-uniqueness" approaches (e.g. Kamp 1981, Heim 1982/1988, 1990, Neale 1990, Kamp & Reyle 1993 among others).

The trade-off is as follows. On the one hand, chapter **5** accounts for a variety of donkey sentences, i.e. cases of intra-sentential anaphora, including mixed reading examples like (96) above. On the other hand, chapter **6** accounts for a variety of uniqueness effects with cross-sentential and intra-sentential anaphora, i.e. forcing the "wide-scope indefinite" reading for discourse (1) above, deriving the relativized uniqueness effects for the "wide-scope indefinite" reading of discourse (2) and deriving the relativized uniqueness effects for the donkey sentence in (91) above.

I will now show that there is a straightforward way to recover the results of chapter **5** within the version of PCDRT introduced in the present chapter. The main observation is that $\mathbf{unique}\{u\}$ conditions are vacuously satisfied under distributivity operators like $\mathbf{dist}_u$, so, to cancel the uniqueness effects, we only need to assume that selective generalized determiners introduce such distributivity operators relative to their nuclear scope update.

The simplest such operator is the unselective distributivity operator defined in (99) below, which is used in the definition of generalized quantification in (103). Note that this definition of generalized quantification differs from the one introduced in (36) above (see section **3.3**) only with respect to the nuclear scope distributivity operator.

99. $\mathbb{dist}(D) := \lambda I_{st}J_{st}.\ \exists R_{s((st)t)}(I{=}\mathbf{Dom}(R) \wedge J{=}\cup\mathbf{Ran}(R) \ \ \wedge \forall{<}i_s{,}J_{st}{>}{\in} R(D\{i\}J)$,

where D is of type $\mathbf{t} := (st)((st)t)$.

100. $\mathbb{dist}_u(D) := \lambda I_{st}J_{st}.\ uI{=}uJ \wedge \forall x_e{\in} uI(\mathbb{dist}(D)I_{u=x}J_{u=x})$

101. $_u\{D\} := \lambda I_{st}.\lambda J_{st}.\ I_{u=\#}{=}J_{u=\#} \wedge I_{u\neq\#}{\neq}\emptyset \wedge \mathbb{dist}_u(D)I_{u\neq\#}J_{u\neq\#}$

102. $_{\langle u\rangle}\{D\} := \lambda I_{st}.\lambda J_{st}.\ I_{u=\#}{=}J_{u=\#} \wedge (I_{u\neq\#}{=}\emptyset \rightarrow I{=}J) \wedge (I_{u\neq\#}{\neq}\emptyset \rightarrow \mathbb{dist}_u(D)I_{u\neq\#}J_{u\neq\#})$

103. $det^{u,u' \sqsubseteq u} \rightsquigarrow \lambda P_{\mathbf{et}}.\lambda P'_{\mathbf{et}}.\ \mathbf{max}^{u}({}_{\langle u \rangle}(P(u)));\ \mathbf{max}^{u' \sqsubseteq u}({}_{\langle u \rangle}\{P'(u')\});\ [\mathbf{DET}\{u, u'\}]$

The distributivity operator dist(*D*) is unselective because any input info state *I* is updated with the DRS *D* in a pointwise manner, i.e. we update each of the 'assignments' $i_s \in I$ with *D*.

This way of updating a set of 'assignments' is unselective in same sense as the generalized quantification over cases proposed in Lewis (1975) is unselective: the definition in (103) instructs us to take each 'assignment' delivered by the restrictor of the quantification separately and check that it satifies the nuclear scope of the quantification, where 'assignments' are also known as: "cases" in the terminology of Lewis (1975), "minimal situations" in the terminology of Heim (1990) and "instances" in the terminology of Kadmon (1990).

Note that the use of unselective distributivity in the definition of dynamic generalized quantification does not endanger our previous results: the definition in (103) does not have a proportion problem (because **DET** relates the relevant sets of invidual) and can account for weak / strong ambiguities, including mixed reading donkey sentences. For example, sentence (96) is represented as shown in (104) below. The **unique**$\{u'\}$ and **unique**$\{u\}$ conditions contributed by the singular donkey pronouns $it_{u'}$ and it_u are vacuously satisfied because the unselective dist operator 'feeds' them only singleton informations states $\{i\}$[35].

104. Every$^{u''}$ person who buys a$^{\mathbf{str}:u}$ TV and has a$^{\mathbf{wk}:u'}$ c.card uses it$_{u'}$ to pay for it$_u$.
$\mathbf{max}^{u''}({}_{u''}([pers\{u''\}];\ \mathbf{max}^{u}([\mathrm{TV}\{u\}, buy\{u'', u\}]);\ [u' \mid c.card\{u'\}, hv\{u'', u'\}]));$
${}_{u''}\{[\mathbf{unique}\{u'\}, \mathbf{unique}\{u\}, {}_{u'}(use\_to\_pay\{u'', u', u\})]\}$

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[35] More precisely: the second conjunct of the definitions of **unique**$\{u'\}$ or **unique**$\{u\}$ is indeed vacuously satisfied with respect to any singleton info state $\{i\}$ whatsoever, but the first conjunct of their definitions fails for any $\{i\}$ such that ui=# or $u'i$=# (i.e. $\{i\}_{u' \neq \#} \neq \emptyset$ is false or $\{i\}_{u \neq \#} \neq \emptyset$ is false). Therefore, the nuclear scope update ${}_{u''}\{[\mathbf{unique}\{u'\}, \mathbf{unique}\{u\}, {}_{u'}(use\_to\_pay\{u'', u', u\})]\}$ fails for any input info state *I* where there is at least one $i_s \in I_{u'' \neq \#}$ such that ui=# or $u'i$=#. But this does not affect the truth-conditions derived by the representation in (104), which are the intuitively correct ones for mixed reading donkey sentence in (96). And, as far as the anaphoric potential goes (both with respect to value and with respect to structure), it seems to me that the above mentioned consequence of the ${}_{u''}\{\ldots\}$ operator is in fact desirable.

Thus, our strategy is to neutralize the uniqueness effects associated with intra-sentential singular anaphora by introducing suitable distributivity operators that take the nuclear scope of the main generalized quantification as argument. This (as opposed to, for example, making the **unique** condition contributed by the singular pronoun optional) has the desirable consequence that we leave untouched the uniqueness effects associated with cross-sentential anaphora in general and with quantificational subordination in particular; that is, we preserve all the results previously obtained in this chapter (see sections **4** and **6.1** above).

Summarizing, the increased flexibility of the theoretical architecture of PCDRT (when compared to previous approaches) enables it to account for the unstable uniqueness intuitions associated with donkey anaphora. The account makes crucial use of plural info states and distributivity operators. More precisely, in any tripartite quantificational structure, we have a choice between selective and unselective nuclear scope distributivity. The decision to use one or the other depends on how we 'think about' the relation between the restrictor and the nuclear scope of the quantification on a particular occasion (which, in turn, is determined by the global discourse context, world knowledge etc., i.e. by various pragmatic factors):

- if we focus on the individuals contributed by the restrictor, we predicate the nuclear scope of each such individual separately, so we use a selective distributivity operator $\mathbf{dist}_u$ and we obtain uniqueness effects (relativized to u);
- if we focus on the (minimal) cases / situations contributed by the restrictor, we predicate the nuclear scope of each such case separately, so we use an unselective distributivity operator $\mathbb{dist}$ and we neutralize / cancel all uniqueness effects.

These two choices, i.e. $\mathbf{dist}_u$ and $\mathbb{dist}$, are the two extremes of a possibly much richer spectrum: if we use $\mathbf{dist}_u$, we are as coarse-grained as possible when we predicate the nuclear scope update; if we use $\mathbb{dist}$,, we are as fine-grained as possible. In between these extremes, we can define a family of of *multiply selective* distributivity operators as shown in (105) and (106) below (see also appendix **0** below). I leave their investigation for future research[36].

105. $_{u,u',\ldots}(D) := \lambda I_{st}.\lambda J_{st}.\ (I_{u=\#}=J_{u=\#} \wedge I_{u'=\#}=J_{u'=\#} \wedge \ldots) \wedge\ I_{u\neq\#,u'\neq\#,\ldots}\neq\emptyset \wedge$

$$\mathbf{dist}_{u,u',\ldots}(D)I_{u\neq\#,u'\neq\#,\ldots}J_{u\neq\#,u'\neq\#,\ldots}$$

106. $_{\langle u,u',\ldots\rangle}(D) := \lambda I_{st}.\lambda J_{st}.\ (I_{u=\#}=J_{u=\#} \wedge I_{u'=\#}=J_{u'=\#} \wedge \ldots) \wedge (I_{u\neq\#,u'\neq\#,\ldots}=\emptyset \rightarrow I=J) \wedge$

$$(I_{u\neq\#,u'\neq\#,\ldots}\neq\emptyset \rightarrow \mathbf{dist}_{u,u',\ldots}(D)I_{u\neq\#,u'\neq\#,\ldots}J_{u\neq\#,u'\neq\#,\ldots})$$

6.3. Telescoping

The phenomenon of telescoping is exemplified by discourses (107) and (108) below, where a singular pronoun seems to be cross-sententially anaphoric to a quantifier. The term is due to Roberts (1987, 1989) and is meant to capture the fact that, in such discourses, "from a discussion of the general case, we zoom in to examine a particular instance" (Roberts 1987: 36).

107. **a.** Each^{u} candidate for the space mission meets all our requirements.
b. He$_{u}$ has a PhD in Astrophysics and extensive prior flight experience.
(Roberts 1987: 36, (38)[37])

108. **a.** Each^{u} degree candidate walked to the stage.
b. He$_{u}$ took his$_{u}$ diploma from the Dean and returned to his$_{u}$ seat.
(Roberts 1987: 36, (34), attributed to B. Partee)

The main observation about this phenomenon (which can be traced back to a pair of examples due to Fodor & Sag 1982 and to Evans 1980) is that "the possibility of anaphoric relations in such telescoping cases depends in part on the plausibility of some sort of narrative continuity between the utterances in the discourse" (Roberts 1987: 36). Thus, Evans (1980) observes that the discourse in (109) below is infelicitous. The examples in (110) and (111) from Poesio & Zucchi (1992) are similarly infelicitous.

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[36] The analysis of the interaction between donkey anaphora and quantificational adverbs (*always*, *usually* etc.) in conditionals might require such multiply selective distributivity operators.

[37] Page references to Roberts (1990).

109. #Every^u congressman came to the party and he$_u$ had a marvelous time.
(Evans 1980: 220, (21)[38])

110. #Every^u dog came in. It$_u$ lay down under the table.
(Poesio & Zucchi 1992: 347, (1))

111. #Each^u dog came in. It$_u$ lay down under the table.
(Poesio & Zucchi 1992: 360, (39c))

The challenge posed by telescoping is to account both for the felicity of (107) and (108) and for the infelicity of (109), (110) and (111), as Poesio & Zucchi (1992) and Roberts (1995, 1996) among others emphasize.

In this respect, DRT / FCS / DPL approaches (Kamp 1981, Heim 1982/1988, Kamp & Reyle 1993 among others) fail because they can account only for the infelicity of (109), (110) and (111), but not for the felicity of (107) and (108). This is a direct consequence of the fact that generalized quantifiers are externally static in this kind of systems (such systems also fail to account for the quantificational subordination discourse in (2) above). Dynamic Montague Grammar (DMG, see Groenendijk & Stokhof 1990) and systems based on it (e.g. Dekker 1993) define generalized quantifers as externally dynamic and, therefore, fail in the opposite way: they can account for the felicity of (107) and (108), but not for the infelicity of (109), (110) and (111). Moreover, DMG does not derive the correct truth-conditions for all telescoping and quantificational subordination discourses (see the discussion in Poesio & Zucchi (1992): 357-359).

The analyses of telescoping in Poesio & Zucchi (1992), Roberts (1995, 1996)[39] and Wang et al (2006) (among others – see the detailed discussion in Wang et al 2006) are more flexible and they can account for both kinds of examples. These accounts make crucial use of more general, pragmatic notions having to with world knowledge and global discourse structure: (**i**) accommodation (for Poesio & Zucchi 1992 and Roberts 1995, 1996) and (**ii**) rhetorical relations (for Wang et al 2006). These accounts differ with

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[38] Page references to Evans (1980).

[39] See also the modal subordination accounts in Geurts (1995/1999) and Frank (1996), which could be generalized to quantificational subordination following the same basic strategy as Poesio & Zucchi (1992) and Roberts (1995, 1996).

respect to their main strategy of analysis: Poesio & Zucchi (1992) and Roberts (1995, 1996) take the infelicitous examples as basic and then devise special mechanisms to account for the felicitous examples, which extract and pass on the relevant discourse information; Wang et al (2006) take the felicitous examples as basic, assume that the relevant discourse information is always available, but that it has to *accessed* in a particular way.

The PCDRT account of telescoping I will sketch below falls in the same category as the Wang et al (2006) account: plural information states ensure that the relevant information is always available, but the singular number morphology on the anaphoric pronoun constrains the way in which it can be accessed. At the same time, I will make limited use of accommodation – and, in this respect, the account is similar to Poesio & Zucchi (1992) and Roberts (1995, 1996).

The PCDRT account is a development of the suggestion made in Evans (1980): 220 with respect to the infelicity of (109). Evans conjectures that the infelicity is a consequence of a clash in *semantic* number between the antecedent and the anaphor (note that there is no clash in morphological number): on the one hand, the quantificational antecedent contributes a non-singleton condition on its restrictor set; on the other hand, the singular pronoun anaphoric to the restrictor set requires it to be a singleton.

I will formalize the non-singleton requirement contributed by selective generalized determiners by means of the **non-unique** condition defined in (112) below[40].

112. $\textbf{non-unique}\{u\} := \lambda I_{st}.\ I_{u\neq\#}\neq\emptyset \wedge \exists i_s,i'_s\in I_{u\neq\#}(ui\neq ui')$

In addition, I will make use of two ingredients independently motivated by the uniqueness effects associated with donkey anaphora (see the previous section), namely: (**i**) the **unique**$\{u\}$ condition contributed by the singular number morphology on a

[40] Green (1989) and Chierchia (1995) (among others) argue that this non-singleton condition has presuppositional status. In contrast, Neale (1990) suggests that it is in fact an implicature. I find the arguments in Green (1989) more persuasive, but I leave a more careful investigation of this issue for future research. For simplicity, I will take the **non-unique**$\{u\}$ condition contributed by generalized determiners to be part of the assertion.

pronoun and (**ii**) the fact that a $\textbf{dist}_u$ or $\mathbb{dist}$ operator that takes scope over such a condition ensures that it is vacuously satisfied.

More concretely, I will assume that the global accommodation of a distributivity operator $_u(\ldots)$ is licensed in the case of the felicitous example analyzed in (113) below, but it is not licensed in the case of the infelicitous example analyzed in (114).

113. Each^u candidate meets all our requirements. $_u$(He$_u$ has a PhD in Astrophysics).
     $\textbf{max}^u([candidate\{u\}]);\ _u([meet\_requirements\{u\}]);\ [\textbf{non-unique}\{u\}];$
     $_u([\textbf{unique}\{u\},\ have\_PhD\{u\}])$
114. Each^u dog came in. #It$_u$ lay down under the table.
     $\textbf{max}^u([dog\{u\}]);\ _u([come\_in\{u\}]);\ [\textbf{non-unique}\{u\}];$
     $[\textbf{unique}\{u\},\ lay\_under\_table\{u\}]$

Of course, nothing in the above analysis specifies when we can and when we cannot accommodate such a distributivity operator. I will return to this issue below. For now, note only that accommodating such an operator should not come for free because we introduce a new meaning component in the discourse representation that is not associated with any morpho-syntactic realization.

The account of the felicitous example in (113) above captures in a direct way the 'telescoping' intuitions associated with it, i.e. the fact that, as Roberts (1987) puts it, the second sentence "zooms in" from a discussion of the general case to a particular instance: the distributivity operator partitions a particular domain of quantification and each cell of the partition is associated with a particular individual; after the domain is partitioned in this way, we update each cell in the partition separately, i.e. "instance by instance".

Note also that the PCDRT account correctly predicts that telescoping cases with plural pronouns are felicitous (or at least better than their singular counterparts), as shown by (115) and (116) below. The reason is that plural pronouns like *they*$_u$ do not contribute a $\textbf{unique}\{u\}$ condition, hence there is no need to accommodate a distributivity operator $_u(\ldots)$ to neutralize / cancel the effects of such a condition.

115. **a.** Every^u dog came in. **b.** (?)They$_u$ lay down under the table.

116. **a.** Every^u congressman came to the party. **b.** (?)They^u had a marvelous time. (Evans 1980: 220, (22))

Similarly, PCDRT can account for the plural anaphora example in (117), which combines quantificational subordination and telescoping – and, also, for the variation on this example in (118). Note in particular that PCDRT can capture the relativized uniqueness effects in (118), i.e. the fact that, intuitively, every man loves exactly one woman; this is due to the fact that the **unique**$\{u'\}$ condition contributed by the singular pronoun *her*$_{u'}$ is within the scope of the distributivity operator $_u(\ldots)$ contributed by the pronoun *they*$_u$.

117. **a.** Every^u man loves a$^{u'}$ woman. **b.** They$_u$ bring them$_{u'}$ flowers to prove this. (van den Berg 1996a: 168, (16))
118. **a.** Every^u man loves a$^{u'}$ woman. **b.** They$_u$ bring her$_{u'}$ flowers. (Wang et al 2006: 7, (20))

Moreover, PCDRT can capture the relativized uniqueness associated with the cross-sentential anaphora *a*$^{u'}$ *spare pawn-it*$_{u'}$ in example (119) below from Sells (1985) (see also Kadmon 1990 for discussion). We only need to assume that a distributivity operator $_u(\ldots)$ with scope over the second sentence in (119) is accommodated. At the same time, PCDRT correctly predicts that the restrictive relative clause example in (120) (also from Sells 1985) does not have relativized uniqueness implications associated with it.

119. **a.** Every^u chess set comes with a$^{\mathbf{str}:u'}$ spare pawn.
b. $_u($ It$_{u'}$ is taped to the top of the box).
120. Every^u chess set comes with a$^{\mathbf{wk/str}:u'}$ spare pawn that is taped to the top of the box.

Using the same ingredients, PCDRT can also account for the contrast in acceptability between (121) and (122) below, from Roberts (1996) (examples (1) and (1') on p. 216) – we only need to assume that the accommodation of a distributivity operator $_u(\ldots)$ is possible in the case of (122) but not in the case of (121).

121. **a.** Every^u frog that saw an$^{u'}$ insect ate it$_{u'}$. #It$_{u'}$ was a fly.
122. **a.** Every$^{u'}$ frog that saw an$^{u'}$ insect ate it$_{u'}$. **b.** $_u($ It$_{u'}$ disappeared forever).

The infelicity of (121) is derived as follows: given that it is not possible for multiple frogs to eat the same insect (this is world knowledge), after we process the update contributed by the first sentence in (121), there should be at least as many eaten insects as there are frogs. But the **unique**$\{u'\}$ condition contributed by the pronoun $it_{u'}$ in the second sentence of (121), which is not in the scope of any distributivity operator, requires that there is only one such eaten insect (which, by the way, was a fly). Since there are at most as many insects as there are frogs, this means that the set of frogs is (at most) a singleton, which contradicts the **non-unique**$\{u\}$ condition contributed by the determiner $every^u$.

Finally, the same ingredients also enable us to account for the examples in (123) and (124) below from Wang et al (2006) (examples (2) on p. 1 and (19) on p. 7 respectively) – and for the relativized uniqueness effects associated with (123).

123. **a.** Every^u hunter that saw a$^{u'}$ deer shot it$_{u'}$. **b.** $_u$(It$_{u'}$ died immediately.)

124. **a.** Every^u hunter that saw a$^{u'}$ deer shot it$_{u'}$. **b.** They$_{u'}$ died immediately.

The problem left unaddressed by the account sketched above is how to decide when we can and when we cannot accommodate such distributivity operators. – and which distributivity operator it is, i.e. which quantificational domain we "zoom in". PCDRT, which is a semantic framework, does not (have to) say any thing about this – but I want to suggest that it offers the two things that we can expect from a semantic theory, namely: (**i**) it provides a precisely circumscribed way in which a more general pragmatic theory can interface with the semantic theory and (**ii**) when the pragmatic 'parameters' / factors are specified, it delivers the intuitively correct truth-conditions.

The previous literature uncovered two important factors that determine whether a distributivity operator can be accommodated or not in PCDRT: (**i**) the rhetorical structure of the discourse – see Wang et al (2006) and (**ii**) general world knowledge – see the notion of script in Poesio & Zucchi (1992). As I have suggested, PCDRT needs to be supplemented with the same kind of pragmatic theory that these alternative approaches assume; there are, however, certain differences between PCDRT and these alternative approaches.

Compared to the accommodation theories proposed in Poesio & Zucchi (1992) and Roberts (1995, 1996)[41], which involve accommodation of discourse referents, conditions, DRS's etc. (triggered by the presuppositional nature of quantifier domain restriction in the case of Roberts 1995, 1996), the PCDRT accommodation procedure is much simpler and involves a clearly circumscribed alteration of the discourse representation, namely: the global accommodation of a distributivity operator with the purpose of satisfying the **unique**$\{u\}$ presupposition[42] contributed by singular number morphology on pronouns. Therefore, I expect that the over-generation problem faced by PCDRT is milder than the one faced by these theories.

Compared to Wang et al (2006) and van den Berg (1996a) (and also Poesio & Zucchi 1992 and Roberts 1987, 1989, 1995, 1996), PCDRT has the advantage that, given its underlying type logic, a Montagovian compositional interpretation procedure can be easily specified, as the present chapter and the previous one have shown.

Moreover, PCDRT simplifies the system in van den Berg (1996a) both with respect to the underlying logic (which is not partial anymore) and with respect to various definitions (e.g. the definitions of the maximization and distributivity operatos) and translations (e.g. the translation of indefinite articles and pronouns).

Finally, unlike the account proposed in Wang et al (2006) (see p. 17 et seqq), the PCDRT account of telescoping is more modular, in the sense that its semantic interpretation procedure (i.e. type-driven translation) is separated from the more global pragmatics of discourse (which involves world knowledge, rhetorical relations etc.). The separation of the semantic and pragmatic interpretive components in PCDRT enables us to simplify multiple aspects of the semantic theory: its underlying logic, the notion of

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[41] Roberts (1995, 1996) build on the more explicit account in Roberts (1987, 1989), which involves accommodation of DRS's.

[42] The condition **unique**$\{u\}$ contributed by number morphology on pronouns is clearly presuppositional – I treat it as an assertion throughout the present dissertation only for simplicity.

See Heim (2005) and reference therein for more discussion of the presuppositional contributions of pronominal morphology. See Beaver & Zeevat (2006) for a recent discussion of accommodation. See Kramer (1998) for a systematic investigation of presupposition in a framework based on CDRT (Muskens 1996), hence closely related to PCDRT.

info state that we use, the operators that we need to access the information stored in these states and the translations given for various lexical items.

I will conclude with the observation that the brief comparison with the previous literature in the last three sections can only be preliminary – and for at least three reasons:

- uniqueness implications are taken to have presuppositional status in much of the previous literature (and for good reason), while I have assumed (for simplicity) that the **unique**$\{u\}$ condition is part of the assertion; thus, a more thorough comparison will be possible only when PCDRT is extended with a theory of presupposition (see Krahmer 1998 for an extensive investigation of presupposition within a framework that also builds on the CDRT of Muskens 1996);
- the import of various design choices specific to different theoretical architectures can be properly evaluated only in the context of a precise investigation of the factors that affect uniqueness in particular instances of singular intra- and cross-sentential anaphora – and such an investigation is beyond the scope of the present dissertation (but see Roberts 2003 and Wang et al 2006 for two recent discussions);
- the uniqueness implications associated with singular cross-sentential anaphora are closely related to the maximality implications associated with plural cross-sentential anaphora – and a proper comparison needs to take into account how any given theory fares with respect to both of them; the present investigation, however, focuses on *morphologically singular* anaphora and on the arguments it provides for a notion of *plural* information state[43].

Given the primarily foundational purpose of the present investigation, such issues can be addressed only partially – but I hope to have at least shown that PCDRT provides a promising framework within which it is possible to formulate simpler and, in certain respects, better analyses of quantificational subordination, donkey anaphora and telescoping and the uniqueness effects associated with them.

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[43] For more discussion of the distinction between plural information states and morphologically plural anaphora, see chapter **8** below and Brasoveanu 2006c.

Appendix

A1. Extended PCDRT: The New Definitions and Translations

125. **Structured Inclusion, Maximization and Distributivity Operators.**

a. $u' \Subset u := \lambda I_{st}.\ \forall i_s \in I(u'i = ui \vee u'i = \#)$

b. $u' \sqsubseteq u := \lambda I_{st}.\ (u' \Subset u)I \wedge \forall i_s \in I(ui \in u'I_{u' \neq \#} \rightarrow ui = u'i)$

c. $\mathbf{max}^{u}(D) := \lambda I_{st}.\lambda J_{st}.\ ([u];\ D)IJ \wedge \forall K_{st}(([u];\ D)IK \rightarrow uK_{u \neq \#} \subseteq uJ_{u \neq \#})$

d. $\mathbf{max}^{u' \sqsubseteq u}(D) := \mathbf{max}^{u'}([u' \sqsubseteq u];\ D)$

e. $\mathbf{dist}_{u}(D) := \lambda I_{st}.\lambda J_{st}.\ \forall x_e(I_{u=x} \neq \emptyset \leftrightarrow J_{u=x} \neq \emptyset) \wedge \forall x_e(I_{u=x} \neq \emptyset \rightarrow DI_{u=x}J_{u=x})$,

i.e. $\mathbf{dist}_{u}(D) := \lambda I_{st}.\lambda J_{st}.\ uI = uJ \wedge \forall x_e \in uI(DI_{u=x}J_{u=x})$,

where $I_{u=x} := \{i_s \in I\colon ui = x\}$

f. $_{u}(D) := \lambda I_{st}.\lambda J_{st}.\ I_{u=\#} = J_{u=\#} \wedge I_{u \neq \#} \neq \emptyset \wedge \mathbf{dist}_{u}(D)I_{u \neq \#}J_{u \neq \#}$

g. $_{\langle u \rangle}(D) := \lambda I_{st}.\lambda J_{st}.\ I_{u=\#} = J_{u=\#} \wedge (I_{u \neq \#} = \emptyset \rightarrow I = J) \wedge (I_{u \neq \#} \neq \emptyset \rightarrow \mathbf{dist}_{u}(D)I_{u \neq \#}J_{u \neq \#})$

h. $_{u}(C) := \lambda I_{st}.\ I_{u \neq \#} \neq \emptyset \wedge \forall x \in uI_{u \neq \#}(CI_{u=x})$, where C is a condition (of type $(st)t$)

i. $_{u}(u_1, \ldots, u_n) := \lambda I_{st}.\lambda J_{st}.\ I_{u=\#} = J_{u=\#} \wedge I_{u \neq \#}[u_1, \ldots, u_n]J_{u \neq \#}$,

where $u \notin \{u_1, \ldots, u_n\}$ and $[u_1, \ldots, u_n] := [u_1]; \ldots; [u_n]$

126. **Distributivity-based Equivalences.**

a. $_{u}([C_1, \ldots, C_m]) = [_{u}(C_1), \ldots, {}_{u}(C_m)]$

b. $_{u}([u_1, \ldots, u_n \mid C_1, \ldots, C_m]) = [_{u}(u_1, \ldots, u_n) \mid {}_{u}(C_1), \ldots, {}_{u}(C_m)])$

127. **Atomic Conditions.**

a. $R\{u_1, \ldots, u_n\} := \lambda I_{st}.\ I_{u_1 \neq \#, \ldots, u_n \neq \#} \neq \emptyset\ \wedge$

$$\forall i_s \in I_{u_1 \neq \#, \ldots, u_n \neq \#}(R(u_1 i, \ldots, u_n i)),$$

where $I_{u_1 \neq \#, \ldots, u_n \neq \#} := \{i_s \in I: u_1 i \neq \# \wedge \ldots \wedge u_n i \neq \#\}$

b. $\mathbf{DET}\{u, u'\} := \lambda I_{st}.\ \mathbf{DET}(uI_{u\neq\#}, u'I_{u'\neq\#})$, where **DET** is a static determiner.

c. $\mathbf{unique}\{u\} := \lambda I_{st}.\ I_{u\neq\#} \neq \emptyset \wedge \forall i_s, i'_s \in I_{u\neq\#}(ui = ui')$

128. **Translations.**

a. *det*$^{u,u'\sqsubseteq u} \rightsquigarrow \lambda P_{\mathbf{et}}.\lambda P'_{\mathbf{et}}.\ \mathbf{max}^{u}({}_{\langle u\rangle}(P(u)));\ \mathbf{max}^{u'\sqsubseteq u}({}_{\langle u'\rangle}(P'(u')));\ [\mathbf{DET}\{u, u'\}]$

b. *det*$_{u}{}^{u'\sqsubseteq u} \rightsquigarrow \lambda P_{\mathbf{et}}.\ \mathbf{max}^{u'\sqsubseteq u}({}_{\langle u'\rangle}(P(u')));\ [\mathbf{DET}\{u, u'\}]$

c. *a*$^{\mathrm{wk}:u} \rightsquigarrow \lambda P_{\mathbf{et}}.\lambda P'_{\mathbf{et}}.\ [u];\ {}_{u}(P(u));\ {}_{u}(P'(u))$

d. *a*$^{\mathrm{str}:u} \rightsquigarrow \lambda P_{\mathbf{et}}.\lambda P'_{\mathbf{et}}.\ \mathbf{max}^{u}({}_{u}(P(u));\ {}_{u}(P'(u)))$

e. *he*$_{u} \rightsquigarrow \lambda P_{\mathbf{et}}.\ [\mathbf{unique}\{u\}];\ {}_{u}(P(u))$

f. *they*$_{u} \rightsquigarrow \lambda P_{\mathbf{et}}.\ {}_{u}(P(u))$

g. *the_sg*$_{u} \rightsquigarrow \lambda P_{\mathbf{et}}.\lambda P'_{\mathbf{et}}.\ [\mathbf{unique}\{u\}];\ {}_{u}(P(u));\ {}_{u}(P'(u))$

h. *the_pl*$_{u} \rightsquigarrow \lambda P_{\mathbf{et}}.\lambda P'_{\mathbf{et}}.\ {}_{u}(P(u));\ {}_{u}(P'(u))$

i. *the_sg*$^{u} \rightsquigarrow \lambda P_{\mathbf{et}}.\lambda P'_{\mathbf{et}}.\ \mathbf{max}^{u}({}_{u}(P(u)));\ [\mathbf{unique}\{u\}];\ {}_{u}(P'(u))$

j. *the_pl*$^{u} \rightsquigarrow \lambda P_{\mathbf{et}}.\lambda P'_{\mathbf{et}}.\ \mathbf{max}^{u}({}_{u}(P(u)));\ {}_{u}(P'(u))$

k. *the_sg*$_{u}{}^{u'} \rightsquigarrow \lambda P_{\mathbf{et}}.\lambda P'_{\mathbf{et}}.\ {}_{u}(\mathbf{max}^{u'}({}_{u'}(P(u')));\ [\mathbf{unique}\{u'\}];\ {}_{u'}(P'(u')))$

l. *Harvey*$^{u} \rightsquigarrow \lambda P_{\mathbf{et}}.\ [u \mid u \in Harvey];\ {}_{u}(P(u))$, where $Harvey := \lambda i_s.\ harvey_e$

m. *every*$^{u} \rightsquigarrow \lambda P_{\mathbf{et}}.\lambda P'_{\mathbf{et}}.\ \mathbf{max}^{u}\ ({}_{u}(P(u)));\ {}_{u}(P'(u))$

n. *always*$_{u} \rightsquigarrow \lambda P_{\mathbf{et}}.\ {}_{u}(P(u))$

A2. Generalized Selective Distributivity

First, we need to generalize our abbreviation for partition cells induced by dref's over plural information states, as shown in (129) below.

129. $I_{u=x} := \{i_s \in I_{st}: ui=x\}$ and $I_{p=w} := \{i_s \in I_{st}: pi=w\}$.

In general:

$I_{\alpha_1=f_1,\ldots,\alpha_n=f_n} := \{i_s \in I_{st}: \alpha_1 i=f_1 \wedge \ldots \wedge \alpha_n i=f_n\}$,

where the types of the terms $\alpha_1,\ldots,\alpha_n$ are in **DRefTyp** and

for each m s.t. $1 \leq m \leq n$, if the type of α_m is $(s\tau)$, then f_m is of type $\tau \in$ **STyp**.

Second, we generalize DRS-level distributivity to multiple dref's, as shown in (130) below.

130. **DRS-level selective distributivity (i.e. distributivity over type t** $:= (st)((st)t)$**).**

$\mathbf{dist}_u(D) := \lambda I_{st}J_{st}.\ uI=uJ \wedge \forall x_e \in uI(DI_{u=x}J_{u=x})$,

where u is of type $\mathbf{e} := se$ and D is of type $\mathbf{t} := (st)((st)t)$.

In general:

$\mathbf{dist}_{\alpha_1,\ldots,\alpha_n}(D) := \lambda I_{st}J_{st}.\ (\alpha_1 I=\alpha_1 J \wedge \ldots \wedge \alpha_n I=\alpha_n J) \wedge$

$\forall f_1 \in \alpha_1 I \ldots \forall f_n \in \alpha_n I(I_{\alpha_1=f_1,\ldots,\alpha_n=f_n} \neq \emptyset \leftrightarrow J_{\alpha_1=f_1,\ldots,\alpha_n=f_n} \neq \emptyset) \wedge$

$\forall f_1 \in \alpha_1 I \ldots \forall f_n \in \alpha_n I(I_{\alpha_1=f_1,\ldots,\alpha_n=f_n} \neq \emptyset \rightarrow DI_{\alpha_1=f_1,\ldots,\alpha_n=f_n} J_{\alpha_1=f_1,\ldots,\alpha_n=f_n})$,

where the types of the terms $\alpha_1, \ldots, \alpha_n$ are in **DRefTyp**

and for each m s.t. $1 \leq m \leq n$, if the type of α_m is $(s\tau)$, then f_m is of type $\tau \in$ **STyp**

and D is of type $\mathbf{t} := (st)((st)t)$.

The general version of DRS-level selective distributivity is more complicated because we work simultaneously with n partitions induced by the drefs $\alpha_1, \ldots, \alpha_n$ on both the input state I and the output state J. The intersection of two partitions is another partition, but we are not guaranteed than the intersection of any two cells in the two partitions is non-empty – hence the antecedent of the conditional in the third conjunct of the generalized definition in (130), i.e. $I_{\alpha_1=f_1,\ldots,\alpha_n=f_n}\neq\emptyset$.

Moreover, we want to ensure that there is a bijection between the intersection of the n partitions over the input state I and the intersection of the n partitions over the output state J, hence the first two conjuncts in the generalized definition in (130): the first one ensures that the values of the n drefs that we distribute over are the same; the second conjunct ensures that there is a bijection between the non-empty, n-distributive cells in the input state partition and the non-empty, n-distributive cells in the output state partition.

Note that the first two conjuncts in the generalized definition in (130) could be replaced with the biconditional $\forall f_1\ldots\forall f_n(I_{\alpha_1=f_1,\ldots,\alpha_n=f_n}\neq\emptyset \leftrightarrow J_{\alpha_1=f_1,\ldots,\alpha_n=f_n}\neq\emptyset)$, which would make clear the parallel between the general case $\mathbf{dist}_{\alpha_1,\ldots,\alpha_n}(D)$ and the special case $\mathbf{dist}_u(D)$ – since the first conjunct of the special case definition in (130) can be replaced with $\forall x_e(I_{u=x}\neq\emptyset \leftrightarrow J_{u=x}\neq\emptyset)$[44]. We can now easily see that the identity in (131) below holds.

[44] Thus, the two most compact (and completely parallel) definitions are:

(i) $\mathbf{dist}_u(D) := \lambda I_{st}J_{st}.\ \forall x_e(I_{u=x}\neq\emptyset \leftrightarrow J_{u=x}\neq\emptyset) \wedge \forall x_e(I_{u=x}\neq\emptyset \rightarrow DI_{u=x}J_{u=x})$

(ii) $\mathbf{dist}_{\alpha_1,\ldots,\alpha_n}(D) := \lambda I_{st}J_{st}.\ \forall f_1\ldots\forall f_n(I_{\alpha_1=f_1,\ldots,\alpha_n=f_n}\neq\emptyset \leftrightarrow J_{\alpha_1=f_1,\ldots,\alpha_n=f_n}\neq\emptyset) \wedge$
$\forall f_1\ldots\forall f_n(I_{\alpha_1=f_1,\ldots,\alpha_n=f_n}\neq\emptyset \rightarrow DI_{\alpha_1=f_1,\ldots,\alpha_n=f_n}J_{\alpha_1=f_1,\ldots,\alpha_n=f_n})$

131. $\mathbf{dist}_{\alpha}(\mathbf{dist}_{\alpha'}(D)) = \mathbf{dist}_{\alpha,\alpha'}(D)$[45]

(in more detail: $\mathbf{dist}_{\alpha}(\mathbf{dist}_{\alpha'}(D)) = \mathbf{dist}_{\alpha,\alpha'}(D) = \mathbf{dist}_{\alpha',\alpha}(D) = \mathbf{dist}_{\alpha'}(\mathbf{dist}_{\alpha}(D)))$

Finally, we define *generalized* selective distributivity, i.e. distributivity generalized to arbitrary distributable types as shown in (132) below. The distributable types are the same as the dynamically conjoinable types **DCTyp** (see definition (62) in section **4** of chapter **5**).

132. **Generalized Selective Distributivity.**

For any term β of type τ, for any $\tau \in$ **DCTyp**:

$${}_{\delta:\{\alpha_1,\ldots,\alpha_n\}}\beta := \mathbf{dist}_{\alpha_1,\ldots,\alpha_n}(\beta)$$

if τ=**t** and the types of the terms $\alpha_1,\ldots,\alpha_n$ are in **DRefTyp**.

$${}_{\delta:\{\alpha_1,\ldots,\alpha_n\}}\beta := \lambda v_{n+1}.\ {}_{\delta:\{\alpha_1,\ldots,\alpha_n,v_{n+1}\}}\beta(v_{n+1})$$

if $\tau=(\sigma\rho)$, v_{n+1} is of type σ and $\sigma \in$ **DRefTyp**.

Abbreviation. ${}_{\delta:\emptyset}\beta := {}_{\delta}\beta$

To understand the intuition behind the above definition of generalized distributivity, we need to begin with the end, i.e. with the abbreviation. Let us assume that our term β is a dynamic property $P_{\mathbf{et}}$, i.e. an object that can be an argument for an extensional generalized determiner. We want to distribute over this property P, i.e. we want to define a *distributed* property ${}_{\delta}P$ of type **et** based on property P.

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[45] Proof. I use the definitions of $\mathbf{dist}_{\alpha}(D)$ and $\mathbf{dist}_{\alpha,\alpha'}(D)$ in the immediately preceding footnote.

$\mathbf{dist}_{\alpha}(\mathbf{dist}_{\alpha'}(D))IJ = \forall f(I_{\alpha=f}\neq\emptyset \leftrightarrow J_{\alpha=f}\neq\emptyset) \wedge \forall f(I_{\alpha=f}\neq\emptyset \rightarrow \mathbf{dist}_{\alpha'}(D)I_{\alpha=f}J_{\alpha=f}) =$

(since $\mathbf{dist}_{\alpha'}(D)I_{\alpha=f}J_{\alpha=f} = \forall f'(I_{\alpha=f,\alpha'=f'}\neq\emptyset \leftrightarrow J_{\alpha=f,\alpha'=f'}\neq\emptyset) \wedge \forall f'(I_{\alpha=f,\alpha'=f'}\neq\emptyset \rightarrow DI_{\alpha=f,\alpha'=f'}J_{\alpha=f,\alpha'=f'}))$

$\forall f(I_{\alpha=f}\neq\emptyset \leftrightarrow J_{\alpha=f}\neq\emptyset) \wedge \forall f(I_{\alpha=f}\neq\emptyset \rightarrow \forall f'(I_{\alpha=f,\alpha'=f'}\neq\emptyset \leftrightarrow J_{\alpha=f,\alpha'=f'}\neq\emptyset) \wedge \forall f'(I_{\alpha=f,\alpha'=f'}\neq\emptyset \rightarrow DI_{\alpha=f,\alpha'=f'}J_{\alpha=f,\alpha'=f'})) =$

$\forall f\forall f'(I_{\alpha=f,\alpha'=f'}\neq\emptyset \leftrightarrow J_{\alpha=f,\alpha'=f'}\neq\emptyset) \wedge \forall f(I_{\alpha=f}\neq\emptyset \rightarrow \forall f'(I_{\alpha=f,\alpha'=f'}\neq\emptyset \rightarrow DI_{\alpha=f,\alpha'=f'}J_{\alpha=f,\alpha'=f'}) =$

$\forall f\forall f'(I_{\alpha=f,\alpha'=f'}\neq\emptyset \leftrightarrow J_{\alpha=f,\alpha'=f'}\neq\emptyset) \wedge \forall f\forall f'(I_{\alpha=f,\alpha'=f'}\neq\emptyset \rightarrow DI_{\alpha=f,\alpha'=f'}J_{\alpha=f,\alpha'=f'}) = \mathbf{dist}_{\alpha,\alpha'}(D)IJ$. □

By the second clause of definition (132), we have that:

133. $\delta P = \delta_{:\emptyset}P = \lambda v_{\mathbf{e}}.\ \delta_{:\{v\}}P(v)$, where $P(v)$ is a DRS.

Since $P(v)$ is a DRS, i.e. of type **t**, we can apply the first clause of definition (132). Therefore:

134. $\delta P = \delta_{:\emptyset}P = \lambda v_{\mathbf{e}}.\ \delta_{:\{v\}}P(v) = \lambda v_{\mathbf{e}}.\ \mathbf{dist}_v(P(v))$

Thus, the distributed property δP is obtained by distributing over the DRS $P(v)$ with respect to the dref variable v. For example, if we distribute over the extensional properties denoted by *man* and *leave*, we obtain the distributed properties in (135) below.

135. $\delta man = \delta(\lambda v_{\mathbf{e}}.\ [man_{et}\{v\}]) = \lambda v_{\mathbf{e}}.\ \mathbf{dist}_v([man_{et}\{v\}])$

$\delta leave = \delta(\lambda v_{\mathbf{e}}.\ [leave_{et}\{v\}]) = \lambda v_{\mathbf{e}}.\ \mathbf{dist}_v([leave_{et}\{v\}])$

A3. DRS-Level Selective Distributivity: Formal Properties

This appendix investigates the basic formal properties of DRS-level selective distributivity. Crucially, I will assume throughout this chapter the simpler PCDRT system introduced in chapter **5** that does not countenance the dummy individual #. The simpler PCDRT system assigns semantic values to atomic conditions, DRS's etc. that are formally much better behaved than the ones assigned by the PCDRT system of chapter **6** which has to introduce the dummy individual # in order to define structured inclusion.

Let us first define what it means for a DRS D to be closed under arbitrary unions.

136. The union $\cup\mathbb{D}$ of a set $\mathbb{D}$ of pairs of info states $<I, J>$ is defined as the pair of info states $<\cup\mathbf{Dom}(\mathbb{D}), \cup\mathbf{Ran}(\mathbb{D})>$.
137. A DRS D (of type $\mathbf{t} := (st)((st)t)$) is *closed under arbitrary unions* iff, given a set $\mathbb{D}$ of info state pairs s.t. $\mathbb{D}\subseteq D$, we have that $D(\cup\mathbf{Dom}(\mathbb{D}))(\cup\mathbf{Ran}(\mathbb{D}))$, i.e. $\cup\mathbb{D}\in D$.

The following kinds of DRS's are closed under arbitrary unions – again, if we assume their simpler definitions according to the PCDRT system of chapter **5** that does not countenance the dummy individual:

138. **a.** Tests that contain only conditions denoting c-ideals (e.g. atomic conditions, dynamic negations of DRS's whose domains are c-ideals etc.) are closed under arbitrary unions since c-ideals are closed under arbitrary unions.

b. A DRS D of the form $[u_1, \dots, u_n \mid C_1, \dots, C_m]$, where the conditions $C_1, \dots, C_m$ are c-ideals, is closed under arbitrary unions[46].

c. A DRS $\mathbf{max}^u(D)$, where D is of the form $[u_1, \dots, u_n \mid C_1, \dots, C_m]$ and the conditions $C_1, \dots, C_m$ are c-ideals, is closed under arbitrary unions[47].

---

[46] Proof. Recall that the denotation of a DRS D of the form $[u_1, \dots, u_n \mid C_1, \dots, C_m]$, where the conditions $C_1, \dots, C_m$ are c-ideals, can be defined as shown in (ii) below based on the relation in (i).

(i) $\mathbb{R}^D := \{<i_s, j_s>: i[u_1, \dots, u_n]j \wedge j \in (\cup C_1) \cap \dots \cap (\cup C_m)\}$;

(ii) $D = \{<I_{st}, J_{st}>: \exists \mathbb{R}_{s(st)} \neq \emptyset (I = \mathbf{Dom}(\mathbb{R}) \wedge J = \mathbf{Ran}(\mathbb{R}) \wedge \mathbb{R} \subseteq \mathbb{R}^D)\}$.

Now take an arbitrary set $\mathbb{D}$ of info state pairs s.t. $\mathbb{D} \subseteq D$. For any pair of info states $<I, J> \in \mathbb{D}$, there is some $\mathbb{R} \subseteq \mathbb{R}^D$ s.t. $I = \mathbf{Dom}(\mathbb{R})$ and $J = \mathbf{Ran}(\mathbb{R})$. If we take the union of all such relations $\mathbb{R}$, we will obtain a relation $\mathbb{R}^*$ s.t. $\mathbb{R}^* \subseteq \mathbb{R}^D$ and s.t. $\cup \mathbf{Dom}(\mathbb{D}) = \mathbf{Dom}(\mathbb{R}^*)$ and $\cup \mathbf{Ran}(\mathbb{D}) = \mathbf{Ran}(\mathbb{R}^*)$. Hence, we have that $D(\cup \mathbf{Dom}(\mathbb{D}))(\cup \mathbf{Ran}(\mathbb{D}))$. □

[47] Proof. Consider a DRS of the form $\mathbf{max}^u(D)$, where D is of the form in the immediately preceding proof. Then the DRS $D' = ([u]; D) = [u, u_1, \dots, u_n \mid C_1, \dots, C_m]$ is of the same form and has a similar kind of denotation in terms of the relation $\mathbb{R}^{D'}$ defined in (i) below.

(i) $\mathbb{R}^{D'} := \{<i_s, j_s>: i[u, u_1, \dots, u_n]j \wedge j \in (\cup C_1) \cap \dots \cap (\cup C_m)\}$

Note that, in this case, the following identities hold: $\mathbf{Dom}(\mathbf{max}^u(D)) = \mathbf{Dom}([u]; D) = \mathbf{Dom}(D')$ – because, for any info state $I \in \mathbf{Dom}([u]; D)$, there is a maximal state J in the set of output states $([u]; D)I$: this maximal state is the image of I under the relation $\mathbb{R}^{D'}$; since J is the supremum info state, it follows that uJ is also the supremum set of individuals.

Now take an arbitrary set $\mathbb{D}$ of info state pairs s.t. $\mathbb{D} \subseteq \mathbf{max}^u(D)$. We show that $\mathbf{max}^u(D)(\cup \mathbf{Dom}(\mathbb{D}))(\cup \mathbf{Ran}(\mathbb{D}))$, i.e.: (i) $D'(\cup \mathbf{Dom}(\mathbb{D}))(\cup \mathbf{Ran}(\mathbb{D}))$ and (ii) $\forall K(D'(\cup \mathbf{Dom}(\mathbb{D}))K \rightarrow uK \subseteq u(\cup \mathbf{Ran}(\mathbb{D})))$.

We know that $\mathbf{max}^u(D) \subseteq D'$, therefore $\mathbb{D} \subseteq D'$ and (i) follows because D' is closed under arbitrary unions (by the previous proof).

Now suppose (ii) does not hold, i.e. there is a K s.t. $D'(\cup \mathbf{Dom}(\mathbb{D}))K$ and s.t. $uK \not\subseteq u(\cup \mathbf{Ran}(\mathbb{D}))$. Based on the observation above, the set of output states corresponding to $\cup \mathbf{Dom}(\mathbb{D})$, i.e. the set $D'(\cup \mathbf{Dom}(\mathbb{D}))$, has a supremum info state, i.e. the image of $\cup \mathbf{Dom}(\mathbb{D})$ under the relation $\mathbb{R}^{D'}$. Let's abbreviate it as $\mathbb{J}^*$. Now, since $\mathbb{J}^*$ is the supremum info state, the set $u\mathbb{J}^*$ is also the supremum set of individuals, so $uK \subseteq u\mathbb{J}^*$ and, therefore, $u\mathbb{J}^* \not\subseteq u(\cup \mathbf{Ran}(\mathbb{D}))$.

I will now show that $u\mathbb{J}^* = u(\cup \mathbf{Ran}(\mathbb{D}))$, which yields a contradiction. Consider an arbitrary pair of info states $<I, J> \in \mathbb{D}$; given that $\mathbb{D} \subseteq \mathbf{max}^u(D)$, we have that $\mathbf{max}^u(D)IJ$, i.e. that $\forall K(D'IK \rightarrow uK \subseteq uJ)$. In

d. A DRS D; D' is closed under arbitrary unions if D and D' are closed under arbitrary unions, i.e. dynamic conjunction preserves closure under arbitrary unions[48].

e. A DRS $\mathbf{dist}_\alpha(D)$ is closed under arbitrary unions for any dref α if D is closed under arbitrary unions[49].

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particular, we have that $uJ = u\mathbb{J}^I$, where $\mathbb{J}^I$ is the image of I under the relation $\mathbb{R}^{D'}$, i.e. the supremum output state in the set of output states $D'I$. The union of all such supremum output states $\mathbb{J}^I$ corresponding to some input state $I\in\mathbf{Dom}(\mathbb{D})$ is precisely $\mathbb{J}^*$, i.e. $\mathbb{J}^*=\cup_{I\in\mathbf{Dom}(\mathbb{D})}\mathbb{J}^I$ and, therefore, $u\mathbb{J}^*$ is the union of all the sets $u\mathbb{J}^I$. Thus, we have that $u\mathbb{J}^* = \cup_{I\in\mathbf{Dom}(\mathbb{D})}u\mathbb{J}^I = \cup_{J\in\mathbf{Ran}(\mathbb{D})}uJ = u(\cup_{J\in\mathbf{Ran}(\mathbb{D})}J) = u(\cup\mathbf{Ran}(\mathbb{D}))$. Contradiction. □

48 Proof. Take an arbitrary set $\mathbb{D}$ of info state pairs s.t. $\mathbb{D}\subseteq(D;\ D')$. This means that for any $<I, J>\in\mathbb{D}$, there is an H s.t. DIH and $D'HJ$. For every pair $<I, J>\in\mathbb{D}$, choose two other pairs IH and HJ s.t. DIH and $D'HJ$. Abbreviate the union of all the IH pairs $\mathbb{D}'$ and the union of all the HJ pairs $\mathbb{D}''$. We have that $\mathbf{Dom}(\mathbb{D}) = \mathbf{Dom}(\mathbb{D}')$, $\mathbf{Ran}(\mathbb{D}) = \mathbf{Ran}(\mathbb{D}'')$ and $\mathbf{Ran}(\mathbb{D}') = \mathbf{Dom}(\mathbb{D}'')$.

Since $\mathbb{D}'\subseteq D$ and $\mathbb{D}''\subseteq D'$ and D and D' are closed under arbitrary unions, we have that $D(\cup\mathbf{Dom}(\mathbb{D}'))(\cup\mathbf{Ran}(\mathbb{D}'))$ and $D'(\cup\mathbf{Dom}(\mathbb{D}''))(\cup\mathbf{Ran}(\mathbb{D}''))$. Given that $\mathbf{Ran}(\mathbb{D}') = \mathbf{Dom}(\mathbb{D}'')$, we have that $(D;\ D')(\cup\mathbf{Dom}(\mathbb{D}'))(\cup\mathbf{Ran}(\mathbb{D}''))$, i.e. $(D;\ D')(\cup\mathbf{Dom}(\mathbb{D}))(\cup\mathbf{Ran}(\mathbb{D}))$, i.e. D; D' is closed under arbitrary unions. □

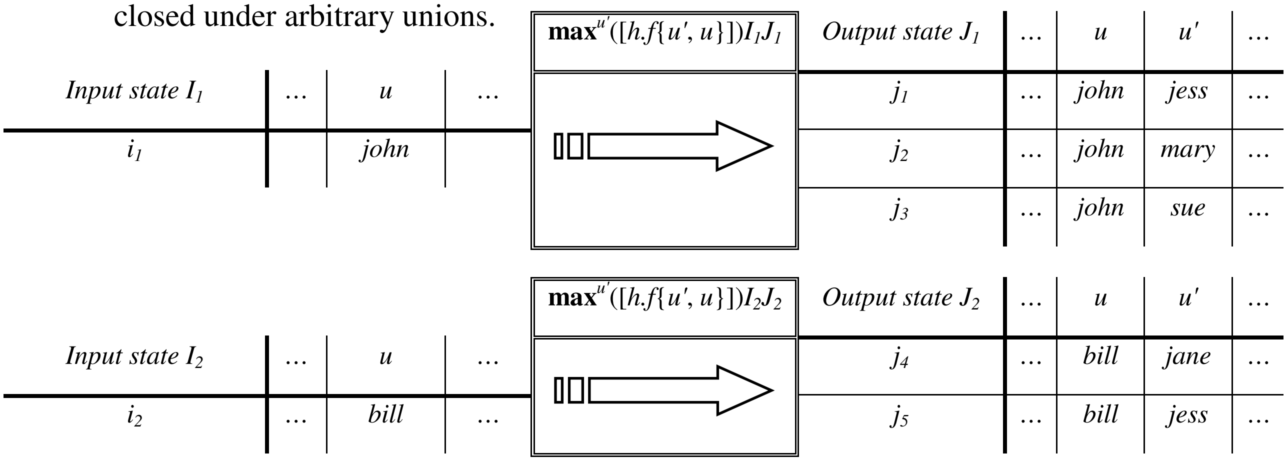
49 Proof. First note that, in general, $\mathbf{dist}_\alpha(D)$ is not closed under arbitrary unions; selective distributivity is based on unions, but *not* on arbitrary unions of info states. Assume, for example, that we have two pairs $<I, J>\in D$ and $<I', J'>\in D$ s.t. $\alpha I=\alpha J$, $\alpha I'=\alpha J'$, $|\alpha I|=|\alpha I'|=1$ and, in addition, $\alpha I=\alpha I'$. Both pairs will be in $\mathbf{dist}_\alpha(D)$, but the union of these two pairs, i.e. $<I\cup I', J\cup J'>$ is not necessarily in the distributed DRS $\mathbf{dist}_\alpha(D)$ – not unless it is in the DRS D itself. This is where the assumption that D is closed under arbitrary unions becomes useful: it entails that $<I\cup I', J\cup J'>\in D$ and, since $\alpha(I\cup I')=\alpha(J\cup J')=\alpha I$ (because we know that $\alpha I=\alpha J=\alpha I'=\alpha J'$) and $|\alpha(I\cup I')|=|\alpha(J\cup J')|=1$, we have that $<I\cup I', J\cup J'>\in\ \mathbf{dist}_\alpha(D)$. The proof generalizes this observation to arbitrary sets of pairs (i.e. there is no more insight to be gained from it). I provide it here for completeness.

Suppose we have a set $\mathbb{D}$ of info states pairs s.t. $\mathbb{D}\subseteq\ \mathbf{dist}_\alpha(D)$. We have to show that $<\cup\mathbf{Dom}(\mathbb{D}), \cup\mathbf{Ran}(\mathbb{D})>\in\mathbf{dist}_\alpha(D)$. By the definition of $\mathbf{dist}_\alpha(D)$, any pair $<I, J>\in\mathbb{D}$ has one of the following two forms: (i) $<I, J>\in D$, $\alpha I=\alpha J$ and $|\alpha I|=1$; (ii) arbitrary unions of sets of pairs of the kind specified in (i), under the condition that, for any two such pairs $<I, J>$ and $<I', J'>$, $\alpha I\neq\alpha I'$. Therefore, $<\cup\mathbf{Dom}(\mathbb{D}), \cup\mathbf{Ran}(\mathbb{D})>$ is the result of taking the union of some arbitrary set of pairs of info states of the kind specified in (i), i.e. pairs of the form $<I, J>$ s.t. $<I, J>\in D$, $\alpha I=\alpha J$ and $|\alpha I|=1$.

We will partition this sets of pairs into equivalence classes as follows: the equivalence class of a given pair $<I, J>$ is the set $\mathbb{D}^{<I,J>}=\{<I', J'>\in D\text{: } \alpha I'=\alpha J' \wedge |\alpha I'|=1 \wedge \alpha I=\alpha I'\}$. For each such equivalence class of pairs $\mathbb{D}^{<I,J>}$, we take its union $\cup\mathbb{D}^{<I,J>}$, where the union is defined as in (136) above, i.e. as the pair of info states $\cup\mathbb{D}^{<I,J>}=<\cup\mathbf{Dom}(\mathbb{D}^{<I,J>}), \cup\mathbf{Ran}(\mathbb{D}^{<I,J>})>$ This pair is in the denotation of the DRS D, i.e. $<\cup\mathbf{Dom}(\mathbb{D}^{<I,J>}), \cup\mathbf{Ran}(\mathbb{D}^{<I,J>})>\in D$, because $\mathbb{D}^{<I,J>}\subseteq D$ and, by assumption, D is closed under arbitrary unions. Moreover, each pair $<\cup\mathbf{Dom}(\mathbb{D}^{<I,J>}), \cup\mathbf{Ran}(\mathbb{D}^{<I,J>})>$ satisfies the conditions $\alpha(\cup\mathbf{Dom}(\mathbb{D}^{<I,J>})=\alpha(\cup\mathbf{Ran}(\mathbb{D}^{<I,J>}))$ (because $\alpha(\cup\mathbf{Dom}(\mathbb{D}^{<I,J>}))=\alpha I=\alpha J=\alpha(\cup\mathbf{Ran}(\mathbb{D}^{<I,J>}))$ and $|\alpha(\cup\mathbf{Dom}(\mathbb{D}^{<I,J>}))|=1$ (because $|\alpha I|=1$). Therefore, for each pair $<I, J>$, we have that $<\cup\mathbf{Dom}(\mathbb{D}^{<I,J>}), \cup\mathbf{Ran}(\mathbb{D}^{<I,J>})>\in\ \mathbf{dist}_\alpha(D)$. Moreover, for any two distinct equivalence classes of pairs $\mathbb{D}^{<I,J>}$ and $\mathbb{D}^{<K,L>}$, their unions $<\cup\mathbf{Dom}(\mathbb{D}^{<I,J>}), \cup\mathbf{Ran}(\mathbb{D}^{<I,J>})>$ and $<\cup\mathbf{Dom}(\mathbb{D}^{<K,L>}), \cup\mathbf{Ran}(\mathbb{D}^{<K,L>})>$ satisfy the additional condition $\alpha(\cup\mathbf{Dom}(\mathbb{D}^{<I,J>}))\neq\alpha(\cup\mathbf{Dom}(\mathbb{D}^{<K,L>}))$. Therefore, the union of such pairs (i.e. of all pairs resulting from unions of equivalence classes) is also in

For example, the DRS $\mathbf{max}^{u'}([happy\_for\{u', u\}])$ is closed under arbitrary unions in the following sense. Suppose that this DRS contains the pairs of info states $<I_1, J_1>$ and $<I_2, J_2>$ in (139) below. The two pairs of info states record the following: given an input state I_1 such that uI_1 is John, the set of individuals that are happy for him are Jessica, Mary and Sue; similarly, given an input state I_2 such that uI_2 is Bill, the set of individuals that are happy for him are Jane and Jessica. Then, the DRS $\mathbf{max}^{u'}([happy\_for\{u', u\}])$ also contains the pair of info states $<I_1 \cup I_2, J_1 \cup J_2>$, since, given the set of individuals $u(I_1 \cup I_2)$, i.e. John and Bill, the set of individuals that are happy for at least one of the two is Jane, Jessica, Mary and Sue.

139. $\mathbf{max}^{u'}([happy\_for\{u', u\}])$ is closed under arbitrary unions.



We can now state the following observation.

140. **Selective distributivity and closure under arbitrary unions.**

If a DRS D is closed under arbitrary unions, then $\mathbf{dist}_\alpha(D) \subseteq D$, for any term α whose type is in **DRefTyp**[50].

$\mathbf{dist}_\alpha(D)$. But this big union is precisely $<\cup\mathbf{Dom}(\mathbb{D}), \cup\mathbf{Ran}(\mathbb{D})>$, i.e. $<\cup\mathbf{Dom}(\mathbb{D}), \cup\mathbf{Ran}(\mathbb{D})> \in \mathbf{dist}_\alpha(D)$ and we have that $\mathbf{dist}_\alpha(D)$ is closed under arbitrary unions. □

[50] Proof. It follows directly from the observation about $\mathbf{dist}_\alpha(D)$ in (i) below and the assumption that D is closed under arbitrary unions. □

(i) The denotation of a DRS $\mathbf{dist}_\alpha(D)$ contains all and only:

(a) those pairs $<I, J> \in D$ such that $\alpha I = \alpha J$ and $|\alpha I| = 1$;

More generally (since $\mathbf{dist}_\alpha(D)$ is closed under arbitrary unions for any dref α if D is closed under arbitrary unions – see (138e) above):

If a DRS D is closed under arbitrary unions, then $\mathbf{dist}_{\alpha_1,\ldots,\alpha_n}(D)\subseteq D$, for any terms $\alpha_1, \ldots, \alpha_n$ whose types are in **DRefTyp**.

The inclusion $\mathbf{dist}_\alpha(D)\subseteq D$ can be strengthened to equality, i.e. we can also show that $D\subseteq\mathbf{dist}_\alpha(D)$, if we require closure under subsets over and above closure under unions.

141. A DRS D (of type $\mathbf{t} := (st)((st)t))$ is *closed under subsets* iff, for any pair of info states $<I, J>\in D$, there is a set $\mathbb{D}\subseteq D$ of info state pairs such that:

(**i**) all the pairs in $\mathbb{D}$ are of the form $<\{i_s\}, \{j_s\}>$, i.e. they contain only singleton info states;

(**ii**) $\cup\mathbb{D}=<I, J>$, where $\cup\mathbb{D}=<\cup\mathbf{Dom}(\mathbb{D}), \cup\mathbf{Ran}(\mathbb{D})>$ (see (136) above), i.e. $I=\cup\mathbf{Dom}(\mathbb{D})$ and $J=\cup\mathbf{Ran}(\mathbb{D})$;

(**iii**) for any set of info state pairs $\mathbb{D}'\subseteq\mathbb{D}$, we have that $D(\cup\mathbf{Dom}(\mathbb{D}'))(\cup\mathbf{Ran}(\mathbb{D}'))$, i.e. $\cup\mathbb{D}'\in D$ (note that this condition follows automatically if D is also closed under unions).

142. The following kinds of DRS's are closed under subsets (if we assume their denotations according to the PCDRT system of chapter **5**):

a. Tests that contain only conditions denoting c-ideals (e.g. atomic conditions, dynamic negations of DRS's whose domains are c-ideals etc.) are closed under subsets since c-ideals are closed under subsets.

b. A DRS D of the form $[u_1, \ldots, u_n \mid C_1, \ldots, C_m]$, where the conditions $C_1, \ldots, C_m$ are c-ideals, is closed under subsets[51].

(b) arbitrary unions of sets of pairs of the kind specified in (a) above, under the condition that, for any two such pairs $<I, J>$ and $<I', J'>$, $\alpha I\neq\alpha I'$ – hence, given that $\alpha I=\alpha J$, $\alpha I'=\alpha J'$ and $|\alpha I|=|\alpha I'|=1$, we have that I and I' are disjoint and J and J' are disjoint.

[51] Proof: Recall that the denotation of a DRS D of the form $[u_1, \ldots, u_n \mid C_1, \ldots, C_m]$, where the conditions $C_1, \ldots, C_m$ are c-ideals, can be defined as shown in (ii) below based on the relation in (i).

(i) $\mathbb{R}^D := \{<i_s, j_s>: i[u_1, \ldots, u_n]j \wedge j\in(\cup C_1)\cap \ldots \cap(\cup C_m)\}$;

We can now state the very useful observation in (143) below, which shows that PCDRT with selective distributivity properly extends the PCDRT system of chapter **5** without selective distributivity only when maximization operators or generalized determiners are involved.

143. **Selective distributivity and closure under arbitrary unions and subsets.**

If a DRS D is closed under arbitrary unions and subsets, then $\mathbf{dist}_\alpha(D)=D$, for any term α whose type is in **DRefTyp** and any DRS D s.t. $\forall\langle I, J\rangle\in D(\alpha I=\alpha J)$[52].

More generally (this follows directly from the special case $\mathbf{dist}_\alpha(D)=D$ and from the fact that $\mathbf{dist}_{\alpha,\alpha'}(D) = \mathbf{dist}_\alpha(\mathbf{dist}_{\alpha'}(D))$ – see (131) above):

If a DRS D is closed under arbitrary unions and under subsets, then $\mathbf{dist}_{\alpha_1,\ldots,\alpha_n}(D)=D$, for any terms $\alpha_1, \ldots, \alpha_n$ whose types are in **DRefTyp** and any DRS D s.t. $\forall\langle I, J\rangle\in D\ (\alpha_1 I=\alpha_1 J \wedge \ldots \wedge \alpha_n I=\alpha_n J)$.

(ii) $D = \{\langle I_{st}, J_{st}\rangle: \exists\mathbb{R}_{s(st)}\neq\emptyset(I=\mathbf{Dom}(\mathbb{R}) \wedge J=\mathbf{Ran}(\mathbb{R}) \wedge \mathbb{R}\subseteq\mathbb{R}^D)\}$.

Now take an arbitrary pair of info states $\langle I, J\rangle\in D$; by (ii), there is some $\mathbb{R}\subseteq\mathbb{R}^D$ s.t. $I=\mathbf{Dom}(\mathbb{R})$ and $J=\mathbf{Ran}(\mathbb{R})$. Take the set $\mathbb{D}$ of info states pairs to be as follows $\mathbb{D} := \{\langle\{i_s\}, \{j_s\}\rangle: \langle i_s, j_s\rangle\in\mathbb{R}\}$. For every pair of info states $\langle\{i\}, \{j\}\rangle\in\mathbb{D}$, there is the singleton relation $\{\langle i, j\rangle\}\subseteq\mathbb{R}\subseteq\mathbb{R}^D$ s.t. $\{i\}=\mathbf{Dom}(\{\langle i, j\rangle\})$ and $\{j\}=\mathbf{Ran}(\{\langle i, j\rangle\})$, therefore $\mathbb{D}\subseteq D$. Moreover, $\cup\mathbb{D} = \langle\cup\mathbf{Dom}(\mathbb{D}), \cup\mathbf{Ran}(\mathbb{D})\rangle = \langle\mathbf{Dom}(\mathbb{R}), \mathbf{Ran}(\mathbb{R})\rangle = \langle I, J\rangle$. The last condition, namely that for any set of info state pairs $\mathbb{D}'\subseteq\mathbb{D}$, it is the case that $D(\cup\mathbf{Dom}(\mathbb{D}'))(\cup\mathbf{Ran}(\mathbb{D}'))$, i.e. that $\cup\mathbb{D}'\in D$, follows directly from (ii), the fact that, for any $\mathbb{R}'\subseteq\mathbb{R}$, $\langle\mathbf{Dom}(\mathbb{R}'), \mathbf{Ran}(\mathbb{R}')\rangle\in D$. □

[52] Proof. Since D is closed under arbitrary unions, we have that $\mathbf{dist}_\alpha(D)\subseteq D$ by observation (140). We just have to prove that $D\subseteq \mathbf{dist}_\alpha(D)$.

Take an arbitrary pair $\langle I, J\rangle\in D$. Since D is closed under subsets, we know that there is a $\mathbb{D}\subseteq D$ s.t. $\cup\mathbb{D} =\langle\cup\mathbf{Dom}(\mathbb{D}), \cup\mathbf{Ran}(\mathbb{D})\rangle = \langle I, J\rangle$ and $\mathbb{D}$ contains only info state pairs of the form $\langle\{i_s\}, \{j_s\}\rangle$. Take a pair $\langle\{i\}, \{j\}\rangle\in\mathbb{D}\subseteq D$. We know that $\alpha\{i\}=\alpha\{j\}$ because, by assumption, $\forall\langle I, J\rangle\in D(\alpha I=\alpha J)$. Moreover, $|\alpha\{i\}|=|\{\alpha i\}|=1$. Therefore, any pair $\langle\{i\}, \{j\}\rangle\in\mathbb{D}$ is s.t. $\langle\{i\}, \{j\}\rangle\in \mathbf{dist}_\alpha(D)$.

We now apply the same technique as the one we used in the proof of (138e). We partition the set $\mathbb{D}$ of pairs into equivalence classes; the equivalence class of a pair $\langle\{i\}, \{j\}\rangle\in\mathbb{D}$ is $\mathbb{D}^{\langle\{i\},\{j\}\rangle} := \{\langle\{i'\}, \{j'\}\rangle\in\mathbb{D}: \alpha i'=\alpha i\}$; thus, $\cup\mathbb{D}^{\langle\{i\},\{j\}\rangle}=\langle\cup\mathbf{Dom}(\mathbb{D}^{\langle\{i\},\{j\}\rangle}), \cup\mathbf{Ran}(\mathbb{D}^{\langle\{i\},\{j\}\rangle})\rangle$ and, since $\mathbb{D} = \cup_{\langle\{i\},\{j\}\rangle\in\mathbb{D}}\mathbb{D}^{\langle\{i\},\{j\}\rangle}$, we have that $\cup\mathbb{D}$, i.e. $\langle I, J\rangle$, is the union of the set of pairs formed $\cup\mathbb{D}^{\langle\{i\},\{j\}\rangle}$.

Since D is closed under arbitrary unions and $\mathbb{D}^{\langle\{i\},\{j\}\rangle}\subseteq\mathbb{D}\subseteq D$, we have that $\cup\mathbb{D}^{\langle\{i\},\{j\}\rangle}=\langle\cup\mathbf{Dom}(\mathbb{D}^{\langle\{i\},\{j\}\rangle}), \cup\mathbf{Ran}(\mathbb{D}^{\langle\{i\},\{j\}\rangle})\rangle\in D$. Moreover, $\alpha(\cup\mathbf{Dom}(\mathbb{D}^{\langle\{i\},\{j\}\rangle})) = \alpha(\{i\}) = \alpha\{j\} = \alpha(\cup\mathbf{Ran}(\mathbb{D}^{\langle\{i\},\{j\}\rangle}))$ and, therefore, we also have that $|\alpha(\cup\mathbf{Dom}(\mathbb{D}^{\langle\{i\},\{j\}\rangle}))|=1$. Thus, $\cup\mathbb{D}^{\langle\{i\},\{j\}\rangle}=\langle\cup\mathbf{Dom}(\mathbb{D}^{\langle\{i\},\{j\}\rangle}), \cup\mathbf{Ran}(\mathbb{D}^{\langle\{i\},\{j\}\rangle})\rangle\in \mathbf{dist}_\alpha(D)$ for any pair $\langle\{i\}, \{j\}\rangle\in\mathbb{D}$. Moreover, since for any two distinct equivalence classes $\mathbb{D}^{\langle\{i\},\{j\}\rangle}$ and $\mathbb{D}^{\langle\{i'\},\{j'\}\rangle}$, we have that $\alpha(\cup\mathbf{Dom}(\mathbb{D}^{\langle\{i\},\{j\}\rangle}))\neq\alpha(\cup\mathbf{Dom}(\mathbb{D}^{\langle\{i'\},\{j'\}\rangle}))$, the union of all $\cup\mathbb{D}^{\langle\{i\},\{j\}\rangle}$ is also in $\mathbf{dist}_\alpha(D)$. But this big union is precisely $\cup\mathbb{D} = \langle\cup\mathbf{Dom}(\mathbb{D}), \cup\mathbf{Ran}(\mathbb{D})\rangle = \langle I, J\rangle$. Thus, $\langle I, J\rangle\in \mathbf{dist}_\alpha(D)$ and therefore $D\subseteq\mathbf{dist}_\alpha(D)$. □

It follows from the observation in (143) above that selective distributive operators are vacuous when applied to tests containing only conditions denoting c-ideals or to DRS's of the form $[u_1, \ldots, u_n \mid C_1, \ldots, C_m]$, where the conditions $C_1, \ldots, C_m$ are c-ideals – if, in the latter case, we distribute over a dref α different from $u_1, \ldots, u_n$.

The equivalence in (144) below shows that, in PCDRT / IP-CDRT, selective distributivity operators distribute over dynamic conjunction.

144. $\mathbf{dist}_\alpha(D; D') = \mathbf{dist}_\alpha(D); \mathbf{dist}_\alpha(D')$,

for any term α whose type is in **DRefTyp** and any DRS's D and D' s.t. $\forall\langle I, J\rangle \in D(\alpha I{=}\alpha J)$ and $\forall\langle I, J\rangle \in D'(\alpha I{=}\alpha J)$[53]

Finally, we show that we can extend our previous results about the reduction of multiply embedded $\mathbf{max}^u$ operators to more complex representations involving selective distributivity in addition to embedded $\mathbf{max}^u$ operators. In particular, the statement in (145) below is a theorem of PCDRT (or IP-CDRT) with selective distributivity. The conditions are identical to the ones needed to reduce structures with multiply embedded $\mathbf{max}^u$ operators that do not contain selectively distributive operators (see the Appendix to the previous chapter).

145. **Simplifying 'Max-under-Max' Representations with selective distributivity:**

$$\mathbf{max}^u(D; \mathbf{dist}_u(\mathbf{max}^{u'}(D'))) = \mathbf{max}^u(D; \mathbf{dist}_u([u']; D')); \mathbf{dist}_u(\mathbf{max}^{u'}(D')),$$

if the following three conditions obtain:

a. u is not reintroduced in D';

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[53] Proof:

$(\mathbf{dist}_\alpha(D); \mathbf{dist}_\alpha(D'))IJ = \exists H(\mathbf{dist}_\alpha(D)IH \wedge \mathbf{dist}_\alpha(D')HJ)$

$= \exists H(\alpha I{=}\alpha H \wedge \forall f{\in}\alpha I(DI_{\alpha=f}H_{\alpha=f}) \wedge \alpha H{=}\alpha J \wedge \forall f{\in}\alpha H(D'H_{\alpha=f}J_{\alpha=f}))$

$= \exists H(\alpha I{=}\alpha H{=}\alpha J \wedge \forall f{\in}\alpha I(DI_{\alpha=f}H_{\alpha=f} \wedge D'H_{\alpha=f}J_{\alpha=f}))$

$=$ (given that $\forall\langle I, J\rangle \in D(\alpha I{=}\alpha J)$ and $\forall\langle I, J\rangle \in D'(\alpha I{=}\alpha J)$) $\alpha I{=}\alpha J \wedge \forall f{\in}\alpha I(\exists H(DI_{\alpha=f}H \wedge D'HJ_{\alpha=f}))$

$= \alpha I{=}\alpha J \wedge \forall f{\in}\alpha I((D; D')I_{\alpha=f}J_{\alpha=f}) = \mathbf{dist}_\alpha(D; D')IJ$. □

b. $\forall I_{st}\forall X_{et}\,(\exists J_{st}(([u']; D')IJ \wedge X{=}uJ) \leftrightarrow \exists J_{st}(\mathbf{max}^{u'}(D')IJ \wedge X{=}uJ))$;

c. $\mathbf{max}^{u'}(D') = [u']; D'; \mathbf{max}^{u'}(D')$[54].

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[54] Proof.

Claim1. If $\forall I_{st}\forall X_{et}(\exists J_{st}(([u']; D')IJ \wedge X{=}uJ) \leftrightarrow \exists J_{st}(\mathbf{max}^{u'}(D')IJ \wedge X{=}uJ))$, then $\forall I_{st}\forall X_{et}(\exists J_{st}(\mathbf{dist}_u([u']; D')IJ \wedge X{=}uJ) \leftrightarrow \exists J_{st}(\mathbf{dist}_u(\mathbf{max}^{u'}(D'))IJ \wedge X{=}uJ))$.

Proof of Claim1. $\forall I_{st}\forall X_{et}(\exists J_{st}(\mathbf{dist}_u([u']; D')IJ \wedge X{=}uJ) \leftrightarrow \exists J_{st}(\mathbf{dist}_u(\mathbf{max}^{u'}(D'))IJ \wedge X{=}uJ))$ =

$\forall I_{st}\forall X_{et}(\exists J_{st}(uI{=}uJ \wedge \forall x_e{\in}\, uI(([u']; D')I_{u=x}J_{u=x}) \wedge X{=}uJ) \leftrightarrow \exists J_{st}(uI{=}uJ \wedge \forall x_e{\in}\, uI(\mathbf{max}^{u'}(D')I_{u=x}J_{u=x}) \wedge X{=}uJ))$ =

$\forall I_{st}(\exists J_{st}(uI{=}uJ \wedge \forall x_e{\in}\, uI(([u']; D')I_{u=x}J_{u=x})) \leftrightarrow \exists J_{st}(uI{=}uJ \wedge \forall x_e{\in}\, uI(\mathbf{max}^{u'}(D')I_{u=x}J_{u=x})))$

LR→: Assume that, for an arbitrary I, we find some J s.t. $uI{=}uJ \wedge \forall x_e{\in}\, uI(([u']; D')I_{u=x}J_{u=x})$. Pick an arbitrary x; we have that $([u']; D')I_{u=x}J_{u=x}$. By hypothesis, we have that $\forall I_{st}\forall X_{et}(\exists J_{st}(([u']; D')IJ \wedge X{=}uJ) \leftrightarrow \exists J_{st}(\mathbf{max}^{u'}(D')IJ \wedge X{=}uJ))$. Instantiate I with $I_{u=x}$ and X with $\{x\}$. We therefore have that:

$\exists J_{st}(([u']; D')I_{u=x}J \wedge \{x\}{=}uJ) \leftrightarrow \exists J_{st}(\mathbf{max}^{u'}(D')I_{u=x}J \wedge \{x\}{=}uJ)$

The left hand-side is true because $([u']; D')I_{u=x}J_{u=x}$ is true and, obviously, $uJ_{u=x}{=}\{x\}$. We can therefore find a state J^x s.t. $\mathbf{max}^{u'}(D')I_{u=x}J^x \wedge \{x\}{=}uJ^x$. Thus, for all $x_e{\in}\, uI$, there is some J^x s.t. $\mathbf{max}^{u'}(D')I_{u=x}J^x \wedge \{x\}{=}uJ^x$. Take the union of all these states, i.e. $\cup_{x\in uI}J^x$. Clearly, $uI{=}u(\cup_{x\in uI}J^x)$ and $\forall x_e{\in}\, uI(\mathbf{max}^{u'}(D')I_{u=x}J^x)$. Therefore $\exists J_{st}(uI{=}uJ \wedge \forall x_e{\in}\, uI(\mathbf{max}^{u'}(D')I_{u=x}J_{u=x}))$.

RL→: the reasoning is parallel. **End of proof of Claim1.**

Thus, $\mathbf{max}^{u}(D; \mathbf{dist}_u(\mathbf{max}^{u'}(D')))IJ$ =

$\exists H(([u]; D)IH \wedge \mathbf{dist}_u(\mathbf{max}^{u'}(D'))HJ) \wedge \forall K(\exists H(([u]; D)IH \wedge \mathbf{dist}_u(\mathbf{max}^{u'}(D'))HK) \rightarrow uK{\subseteq}uJ)$

By condition (145b) and **Claim1**, we have that: $\forall I_{st}\forall X_{et}(\exists J_{st}(_u([u']; D')IJ \wedge X{=}uJ) \leftrightarrow \exists J_{st}(\mathbf{dist}_u(\mathbf{max}^{u'}(D'))IJ \wedge X{=}uJ))$.

Therefore, $\mathbf{max}^{u}(D; \mathbf{dist}_u(\mathbf{max}^{u'}(D')))IJ$ =

$\exists H(([u]; D)IH \wedge \mathbf{dist}_u(\mathbf{max}^{u'}(D'))HJ) \wedge \forall K(\exists H(([u]; D)IH \wedge \mathbf{dist}_u([u']; D')HK) \rightarrow uK{\subseteq}uJ)$ =

$\exists H(([u]; D)IH \wedge \mathbf{dist}_u(\mathbf{max}^{u'}(D'))HJ) \wedge \forall K(([u]; D; \mathbf{dist}_u([u']; D'))IK \rightarrow uK{\subseteq}uJ)$

We have that $\mathbf{max}^{u'}(D') = [u']; D'; \mathbf{max}^{u'}(D')$ (condition (145c)). Hence: $\mathbf{max}^{u}(D; \mathbf{dist}_u(\mathbf{max}^{u'}(D')))IJ$ =

$\exists H(([u]; D)IH \wedge \mathbf{dist}_u([u']; D'; \mathbf{max}^{u'}(D'))HJ) \wedge \forall K(([u]; D; \mathbf{dist}_u([u']; D'))IK) \rightarrow uK{\subseteq}uJ)$

Since u is not reintroduced in D' (condition (145a)), we have by fact (144) that $\mathbf{dist}_u([u']; D'; \mathbf{max}^{u'}(D'))$ = $\mathbf{dist}_u([u']; D'); \mathbf{dist}_u(\mathbf{max}^{u'}(D'))$. Therefore: $\mathbf{max}^{u}(D; \mathbf{dist}_u(\mathbf{max}^{u'}(D')))IJ$ =

$\exists H(([u]; D; \mathbf{dist}_u([u']; D'))IH \wedge \mathbf{dist}_u(\mathbf{max}^{u'}(D'))HJ) \wedge \forall K(([u]; D; \mathbf{dist}_u([u']; D'))IK) \rightarrow uK{\subseteq}uJ)$ =

$\exists H(([u]; D; \mathbf{dist}_u([u']; D'))IH \wedge \forall K(([u]; D; \mathbf{dist}_u([u']; D'))IK) \rightarrow uK{\subseteq}uJ) \wedge \mathbf{dist}_u(\mathbf{max}^{u'}(D'))HJ)$

Since u is not reintroduced in D' (condition (145a)), we have that $uJ{=}uH$. Hence: $\mathbf{max}^{u}(D; \mathbf{dist}_u(\mathbf{max}^{u'}(D')))IJ$ =

$\exists H(([u]; D; \mathbf{dist}_u([u']; D'))IH \wedge \forall K(([u]; D; \mathbf{dist}_u([u']; D'))IK) \rightarrow uK{\subseteq}uH) \wedge \mathbf{dist}_u(\mathbf{max}^{u'}(D'))HJ)$ =

$\exists H(\mathbf{max}^{u}(D; \mathbf{dist}_u([u']; D'))IH \wedge \mathbf{dist}_u(\mathbf{max}^{u'}(D'))HJ) = (\mathbf{max}^{u}(D; \mathbf{dist}_u([u']; D')); \mathbf{dist}_u(\mathbf{max}^{u'}(D')))IJ$. □

Just as before, we can further simplify the three conditions in (145). First, given the first condition, i.e. (145a), the second condition is equivalent to **Dom**([u']; D') = **Dom**(**max**$^{u'}$(D')). Moreover, based on the two facts in (146) (see the appendix of chapter **5** for their proofs), we can further simplify condition (145c).

146. **a.** If D' is of the form [u_1, …, u_n | C_1, …, C_m],

then $\forall I_{st}J_{st}(([u']; D')IJ \rightarrow ([u']; D')I=([u']; D')J)$.

b. If $\forall I_{st}J_{st}(([u']; D')IJ \rightarrow ([u']; D')I=([u']; D')J)$,

then **max**$^{u'}$(D') = [u']; D'; **max**$^{u'}$(D').

Thus, we have the corollary in (147) below.

147. **Simplifying 'Max-under-Max' Representations with selective distributivity (corollary):**

max^{u}(D; **dist**$_u$(**max**$^{u'}$(D'))) = **max**^{u}(D; **dist**$_u$([u']; D')); **dist**$_u$(**max**$^{u'}$(D')),

if the following three conditions obtain:

a. u is not reintroduced in D';

b. **Dom**([u']; D') = **Dom**(**max**$^{u'}$(D'));

c. D' is of the form [u_1, …, u_n | C_1, …, C_m].

The right handside of the identity can be further simplified if the DRS [u']; D' is closed under unions and subsets, in which case we can omit the distributive operator embedded under **max**^{u} since **dist**$_u$([u']; D') = [u']; D' – this holds because, by (147a), u is not reintroduced in D' and, therefore, $\forall\langle I, J\rangle \in ([u']; D')(uI=uJ)$. Consequently, we have the additional corollary in (148) below, which is useful for the simplification of derivations.

148. **Simplifying 'Max-under-Max' Representations with selective distributivity (corollary2):**

max^{u}(D; **dist**$_u$(**max**$^{u'}$(D'))) = **max**^{u}(D; [u']; D'); **dist**$_u$(**max**$^{u'}$(D')),

if the following three conditions obtain:

a. u is not reintroduced in D';

b. **Dom**([u']; D') = **Dom**(**max**$^{u'}$(D'));

c. D' is of the form [u_1, …, u_n | C_1, …, C_m]

and C_1, …, C_m are c-ideals.

Moreover, (148b) actually follows from (148c) because C_1, …, C_m are c-ideals.