Modified Numerals as Post-suppositions

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1 Cumulativity and Modified Numerals

Goal: provide a compositional account of cumulative readings with non-increasing modified numerals (aka van Benthem's puzzle, van Benthem 1986) – exemplified in (1) below.

- we discuss mainly *exactly* n modified numerals, but the same problem arises with other non-increasing numerals, e.g., *at most* n, *up to* n, *as many as* n, *maximally* n etc.

(1) Exactly three^x boys saw exactly five^y movies.

(2) **Figure 1**

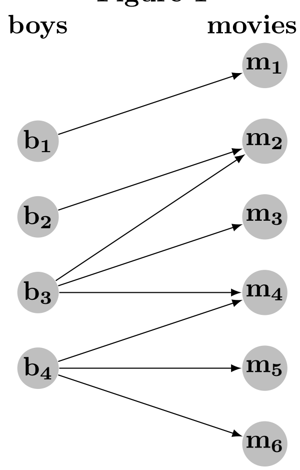
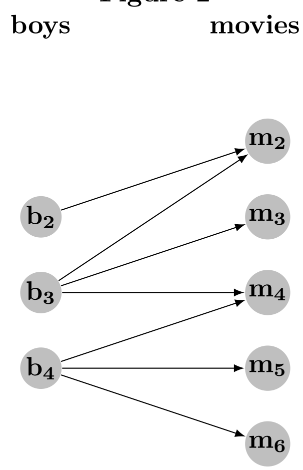


Figure 2



The most salient reading of sentence (1) is the surface-scope distributive one:

- there are exactly 3 boys such that each of them saw exactly 5 movies (possibly different from boy to boy)

We are not interested in this reading (although we discuss it briefly later on).

The reading of sentence (1) that we want to capture is the cumulative reading, namely:

- consider the maximal number of boys that saw a movie and the maximal number of movies seen by a boy
- there are 3 such boys and 5 such movies

Sentence (1) on its cumulative reading could be an exhaustive answer to the question:

(3) How many boys saw how many movies, exactly?

…in a situation like the one in Figure 2 above.

The cumulative reading is different from:[1]

- the maximal number of boys that saw exactly 5 movies is 3

This is actually not a reading of sentence (1), although it bears some resemblance to its distributive reading.

The situations depicted in the Figures 1 and 2 above distinguish between these two readings:

- Figure 1 is exactly like Figure 2, except for the addition of boy $\mathbf{b_1}$, movie $\mathbf{m_1}$ and the arrow between them symbolizing the seeing relation
- the cumulative reading is intuitively false in Figure 1 (4 boys and 6 movies) and true in Figure 2

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[1]As Krifka (1999), Landman (2000) and Ferreira (2007) observe. But see Robaldo (2009) for a different take on the data.

- the second 'reading' is true in both cases

The distinction between the cumulative reading and this other 'reading' is important for theoretical reasons:

- many formal systems derive something like it when they attempt to capture the cumulative reading

- reason: when we interpret sentence (1) compositionally, the maximality condition contributed by the subject *exactly three boys* takes scope of the maximality condition contributed by the direct object *exactly five movies*

- what we want is: simultaneous global maximization over both subject and direct object plus interpreting the cardinality requirements (exactly 3 and exactly 5) outside this maximization

- but it is not obvious how to get this compositionally

Here are some naturally-occurring examples from the Corpus of Contemporary American English (COCA, `www.americancorpus.org`):[2]

(4) CSX and Norfolk Southern, which haul coal from eastern mines, are making lesser but still sizable investments to maintain or upgrade lines that take a beating from coal trains.
Due to their size – up to four locomotives pulling as many as 125 cars – coal trains wear down tracks more quickly than other cargo haulers.

(5) The American people have no idea what the president knows, and that is he's going to have to stay in Iraq with thousands and thousands of troops – as many as estimated, of course, as many as 75,000 troops for up to four or five years.

(6) Yet the fight to recover that money - estimated at up to $10 billion owed to as many as 500,000 Indians - has caused a backlash among some tribal leaders.

(7) That's one of the reasons the Alameda County Board of Supervisors approved a proposed upgrade last December by Green Ridge Power LLC, Altamont Power LLC, Sea West Windfarms Inc. and Ventura Pacific Inc. to replace as many as 1,270 old windmills with up to 187 new ones.

(8) All measurements made with the unit may be stored in the on-board memory that will hold as many as 3,000 readings from up to 100 individual probes.

[2]For more discussion of the less studied construction *as many as n*, see Rett (2009). See Nouwen (2009) and references therein for a detailed discussion of other non-increasing numerals, including *up to*.

(9) In the 10 years to 1987, the city lost as many as 46,000 jobs – worth up to $1.1 billion in lost wages and purchasing power – to corporate consolidations and relocations.

(10) [An example of a distributive reading forced by the addition of an explicit item]
This new organization looks to assist as many as 45 artists per year with loans of up to $3,000 apiece.

Main proposal: modified numerals make two kinds of contributions to the meaning of sentences like (1).

(*i*) their asserted/at-issue contribution is a maximization operator that introduces the maximal set of entities that satisfies their restrictor and nuclear scope

(*ii*) a post-supposition, i.e., a cardinality constraint (e.g., exactly three) that needs to be satisfied relative to the *context that results after* the at-issue meaning is evaluated

Contexts: for our purposes, sets of total variable assignments relative to which quantificational expressions are interpreted – and which are updated as a result of the interpretation of such expressions.

- that is, we work with a simplified version of Dynamic Plural Logic (DPlL, van den Berg 1996)

Using post-suppositions enables us interpret the cardinality requirements *globally* and this is enough:

- we don't need a global maximality operator

- the system is compositional and the maximality conditions are nested, but they are equivalent to a global maximality condition

The main difference between the present account and Krifka (1999) is conceptual: we take modified numerals to constrain *quantificational* – and not focus – alternatives, where a quantificational alternative is one of the contexts satisfying a quantificational expression.

- thus, we reconceptualize DPlL as the logic of quantificational alternatives in natural language interpretation

2 Modified Numerals as Post-suppositions

2.1 Bare Numerals and Singular Indefinites

We work with the usual models for classical first-order logic (FOL) $\mathfrak{M} = \langle \mathfrak{D}, \mathfrak{I} \rangle$:

- $\mathfrak{D}$ is the domain of individuals
- $\mathfrak{I}$ is the basic interpretation function such that $\mathfrak{I}(R) \subseteq \mathfrak{D}^n$ for any n-ary relation R

An $\mathfrak{M}$-assignment g is a total function from the set of variables $\mathcal{V}$ to $\mathfrak{D}$.

Essence of quantification in FOL: pointwise/variablewise manipulation of variable assignments, abbreviated $h[x]g$:

- $h[x]g := h$ differs from g at most with respect to the value it assigns to x

We generalize this to sets of assignments $H[x]G$ cumulative-quantification style:

(11) $H[x]G := \begin{cases} \text{for all } h \in H, \text{ there is a } g \in G \text{ such that } h[x]g \\ \text{for all } g \in G, \text{ there is a } h \in H \text{ such that } h[x]g \end{cases}$

- natural generalization: $H[x]G$ is an equivalence relation, just as $h[x]g$

A set of assignments G can be represented as a matrix:

- the rows of the matrix represent variable assignments g_1, g_2, g_3 etc.
- the columns represent variables x, y etc.
- the objects in the cells of the matrix are values that assignments assign to variables: $boy_1 = g_1(x)$, $boy_2 = g_2(x)$, $movie_1 = g_1(y)$, $movie_2 = g_2(y)$ etc.

(12)

G	...	x	y	...
g_1	...	boy_1	$movie_1$	...
g_2	...	boy_2	$movie_2$	...
g_3	...	boy_3	$movie_3$	...
...	...	...	...	...

or simply:

...	x	y	...
...	boy_1	$movie_1$	...
...	boy_2	$movie_2$	...
...	boy_3	$movie_3$	...
...	...	...	...

Atomic formulas are tests, i.e., they check that the input context G satisfies them and pass this context on:

(13) $[\![R(x_1, \ldots, x_n)]\!]^{\langle G,H \rangle} = \mathbb{T}$ iff $G = H$ and for all $h \in H$, $\langle h(x_1), \ldots, h(x_n)\rangle \in \mathfrak{I}(R)$

Cardinality constraints on the values of variables are also tests:

(14) $G(x) := \{g(x) : g \in G\}$

(15) $|G(x)|$ is the cardinality of the set of individuals $G(x)$

(16) $[\![x = n]\!]^{\langle G,H \rangle} = \mathbb{T}$ iff $G = H$ and $|H(x)| = n$

(17) $[\![x \leq n]\!]^{\langle G,H \rangle} = \mathbb{T}$ iff $G = H$ and $|H(x)| \leq n$

(18) $[\![x \geq n]\!]^{\langle G,H \rangle} = \mathbb{T}$ iff $G = H$ and $|H(x)| \geq n$

Dynamic conjunction and random assignment (DRT/FCS/DPL-style):

(19) $[\![\phi \wedge \psi]\!]^{\langle G,H \rangle} = \mathbb{T}$ iff there is a K s.t. $[\![\phi]\!]^{\langle G,K \rangle} = \mathbb{T}$ and $[\![\psi]\!]^{\langle K,H \rangle} = \mathbb{T}$

(20) Random assignment: $[\![[x]]\!]^{\langle G,H \rangle} = \mathbb{T}$ iff $H[x]G$

The format for the translation of singular indefinite articles and bare numerals:

(21) $\exists x[x = n \wedge \phi]\ (\psi)$ intuitively: n ϕ-individuals are ψ

- square brackets [] indicate restrictor formulas, round brackets ( ) indicate nuclear scope formulas
- singular indefinite article: n is 1
- *A*^{x} *wolf came in* $\rightsquigarrow \exists x[x = 1 \wedge \textsc{wolf}(x)]\ (\textsc{come-in}(x))$
- bare numeral *two*: n is 2
- *Two*^{x} *wolves came in* $\rightsquigarrow \exists x[x = 2 \wedge \textsc{wolf}(x)]\ (\textsc{come-in}(x))$

This translation schema is just an abbreviation:

(22) $\exists x[x = n \wedge \phi]\ (\psi) := [x] \wedge x = n \wedge \phi \wedge \psi$

Proper names are interpreted like indefinites, except their restrictor formula requires the variable to take as its only value the individual that is the bearer of that name:

(23) $\exists x[x = \textsc{jasper}]\ (\phi) := [x] \wedge x = \textsc{jasper} \wedge \phi$

- JASPER is a non-logical constant denoting the individual Jasper

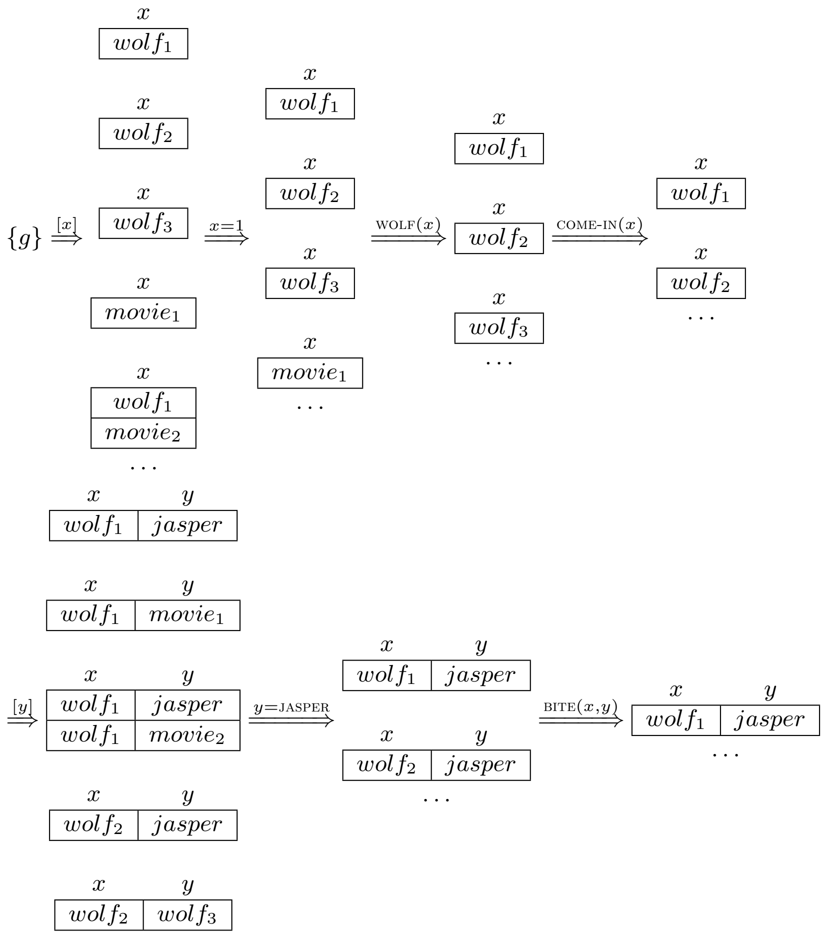
Pronouns are indexed with the variable introduced by their antecedent and their translation is that variable itself; we ignore differences between singular and plural pronouns. For example:

(24) A^x wolf came in. It$_x$ bit Jasper^y.

(25) a. $\exists x[x = 1 \wedge \textsc{wolf}(x)]\ (\textsc{come-in}(x)) \wedge$
$\exists y[y = \textsc{jasper}]\ (\textsc{bite}(x, y))$

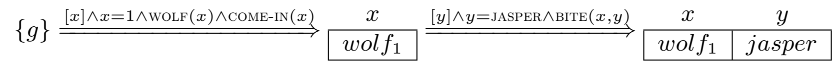
b. $[x] \wedge x = 1 \wedge \text{WOLF}(x) \wedge \text{COME-IN}(x) \wedge$
$[y] \wedge y = \text{JASPER} \wedge \text{BITE}(x, y)$

Suppose, for simplicity, that our input context G is the singleton set $\{g\}$, where g assigns some arbitrary values to all variables. The conjunction of formulas above updates this input context as follows:

(26) 

Except for the fact that we allow matrices with multiple rows, this interpretation graph is in no way different from the way interpretation proceeds in classical first-order logic or in classical DRT/FCS: such graphs are implicit in their recursive definitions of truth and satisfaction.

From now on, we will depict updates by choosing a single, typical path through the graph:

(27) 

The definition of truth below says that a formula is true if there is at least one successful path through the graph denoted by ϕ. Again, this is just as in classical first-order logic or in classical DRT/FCS.

(28) Truth: a formula ϕ is true relative to an input set of assignments G iff there is an output set of assignments H such that $[\![\phi]\!]^{\langle G,H\rangle} = \mathbb{T}$.

Bare numerals are translated and interpreted in a parallel way:

(29) Two^x wolves came in. They$_x$ bit Jasper^y.

(30) a. $\exists x[x = 2 \wedge \text{WOLF}(x)]\ (\text{COME-IN}(x)) \wedge$
$\exists y[y = \text{JASPER}]\ (\text{BITE}(x, y))$

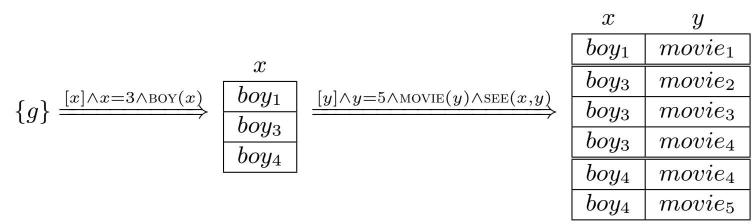
b. $[x] \wedge x = 2 \wedge \text{WOLF}(x) \wedge \text{COME-IN}(x) \wedge$
$[y] \wedge y = \text{JASPER} \wedge \text{BITE}(x, y)$

We get cumulative readings for bare numerals automatically:

(31) Three^x boys saw five^y movies.

(32) a. $\exists x[x = 3 \wedge \text{BOY}(x)]\ (\exists y[y = 5 \wedge \text{MOVIE}(y)]\ (\text{SEE}(x, y)))$

b. $[x] \wedge x = 3 \wedge \text{BOY}(x) \wedge [y] \wedge y = 5 \wedge \text{MOVIE}(y) \wedge \text{SEE}(x, y)$

(33) 

- this is just the matrix representation of part of Figure 1
- note that $\text{BOY}(x)$, $\text{MOVIE}(y)$ and $\text{SEE}(x, y)$ are not cumulatively interpreted (no * or ** operators)
- they are distributively interpreted relative to their input set of assignments, i.e., they relate atomic individuals as usual

We allow for non-atomic individuals as values,[3] e.g., *every three houses*, we just don't need them here. That is, we distinguish between two kinds of pluralities:

- evaluation plurality, i.e., sets of assignments
- domain plurality, i.e., non-atomic individuals

[3]Unlike van den Berg (1996) among others.

2.2 Modified Numerals

We capture the meaning of modified numerals by means of a maximization operator $\mathbf{M}$ that enables us to introduce the set of all individuals that satisfy both the restrictor and nuclear scope of such modified numerals.

(34) $[\![\mathbf{M}(\phi)]\!]^{\langle G,H\rangle} = \mathbb{T}$ iff $[\![\phi]\!]^{\langle G,H\rangle} = \mathbb{T}$ and there is no H' s.t. $H \subsetneq H'$ and $[\![\phi]\!]^{\langle G,H'\rangle} = \mathbb{T}$.

For example, $\mathbf{M}([x] \wedge \textsc{wolf}(x))$ introduces the variable x and requires it to store *all* and *only* the individuals satisfying $\textsc{wolf}(x)$, i.e., the set of wolves.

- $[\![\mathbf{M}([x] \wedge \textsc{wolf}(x))]\!]^{\langle G,H\rangle} = \mathbb{T}$ iff
  - $[\![[x] \wedge \textsc{wolf}(x)]\!]^{\langle G,H\rangle} = \mathbb{T}$: we store in x only individuals that satisfy $\textsc{wolf}(x)$, i.e., x stores only wolves
  - there is no H' s.t. $H \subsetneq H'$ and $[\![[x] \wedge \textsc{wolf}(x)]\!]^{\langle G,H'\rangle} = \mathbb{T}$: there is no way to store in x more individuals and still satisfy $\textsc{wolf}(x)$, i.e., x stores all wolves (maximality)

We can now provide a preliminary translation for modified numerals:

(35) *exactly n* $\quad \exists x = n[\phi]\ (\psi) := \mathbf{M}([x] \wedge \phi \wedge \psi) \wedge x = n$

For example:

- *Exactly three*^{x} *wolves came in*
  $\rightsquigarrow \exists x = 3[\textsc{wolf}(x)]\ (\textsc{come-in}(x))$
  $\rightsquigarrow \mathbf{M}([x] \wedge \textsc{wolf}(x) \wedge \textsc{come-in}(x)) \wedge x = 3$
- intuitively: store in x all the entities that are wolves and that came in; then, test that there are 3 such entities

We can further elaborate on the above sentence with *They*$_x$ *bit Jasper*^{y} and derive intuitively-correct truth conditions for the resulting discourse (modulo 'partial covers').

But we derive incorrect truth conditions for sentence (1):

(36) a. $\exists x = 3[\textsc{boy}(x)]\ (\exists y = 5[\textsc{movie}(x)]\ (\textsc{see}(x,y)))$
b. $\mathbf{M}([x] \wedge \textsc{boy}(x) \wedge \mathbf{M}([y] \wedge \textsc{movie}(y) \wedge \textsc{see}(x,y)) \wedge y = 5) \wedge x = 3$

- we do not derive the cumulative reading, true only in Figure 2
- instead, we derive the 'reading' true in both Figure 1 and Figure 2: the maximal number of boys that saw exactly five movies is three

What we want is a translation that places the cardinality requirement $y = 5$ contributed by the direct object outside the scope of the maximization operator $\mathbf{M}([x] \wedge \dots)$ contributed by the subject:

(37) $\mathbf{M}([x] \wedge \textsc{boy}(x) \wedge \mathbf{M}([y] \wedge \textsc{movie}(y) \wedge \textsc{see}(x,y))) \wedge y = 5 \wedge x = 3$

Equivalently (because the embedded $\mathbf{M}$ operator is vacuous in this case):

(38) $\mathbf{M}([x] \wedge \textsc{boy}(x) \wedge [y] \wedge \textsc{movie}(y) \wedge \textsc{see}(x,y)) \wedge y = 5 \wedge x = 3$

This formula captures the cumulative reading of sentence (1):

- consider the maximal set x of boys that saw a movie and the maximal set y of movies seen by a boy
- there are five such movies and three such boys

2.3 Post-suppositions

To be able to compositionally derive such a representation, we will take cardinality requirements to be part of a dimension of meaning separate from the asserted/at-issue meaning (but closely integrated with it).

We will take them to be *post-suppositions*, i.e., tests on the output context, as opposed to presuppositions, which are tests on the input context. See Lauer (2009) for another use of the same notion and Farkas (2002) and Constant (2006) for related types of post-assertion constraints on output contexts.

Post-suppositions are formulas introduced at certain points in the interpretation that are passed on from local context to local context and that need to be satisfied only globally, relative to the final output context.

- a context is a set of assignments G indexed with a set of tests ζ, represented as $G[\zeta]$
- all the operators above are interpreted in the same way, except that, if the input context G is indexed with a set of tests ζ, this set is passed on to the output state H
- thus, the interpretation function is not simply $[\![\cdot]\!]^{\langle G,H\rangle}$, but $[\![\cdot]\!]^{\langle G[\zeta],H[\zeta']\rangle}$, where ζ and ζ' are sets of tests and $\zeta \subseteq \zeta'$

We mark a formula ϕ as a post-supposition by superscripting it.

(39) $[\![^{\phi}]\!]^{\langle G[\zeta],H[\zeta']\rangle} = \mathbb{T}$ iff ϕ is a test, $G = H$ and $\zeta' = \zeta \cup \{\phi\}$.

- a post-suppositional formula does not update the input set of assignments G
- it is simply added to the set of tests ζ

These tests are post-suppositional in the sense that they are required to be true relative to the final output context (which grants them something very similar to widest scope). This is formalized by means of the definition of truth below.

(40) Truth: a formula ϕ is true relative to an input context $G[\emptyset]$, where $\emptyset$ is the empty set of tests, iff there is an output set of assignments H and a (possibly empty) set of tests $\{\psi_1, \ldots, \psi_m\}$ s.t. $[\![\phi]\!]^{\langle G[\emptyset], H[\{\psi_1, \ldots, \psi_m\}]\rangle} = \mathbb{T}$ and $[\![\psi_1 \wedge \ldots \wedge \psi_m]\!]^{\langle H[\emptyset], H[\emptyset]\rangle} = \mathbb{T}$.

- the definition of truth treats the formulas $\psi_1, \ldots, \psi_m$ as post-suppositions, i.e., as tests performed on the final output set of assignments H (as opposed to presuppositions, i.e., tests performed on input contexts)
- the entire recursive definition of truth and satisfaction needs to be reformulated in terms of sets of assignments indexed with sets of tests $G[\zeta]$ rather than simply sets of assignments G – see the appendix

Modified numerals are interpreted as before, except that the cardinality requirement is a post-supposition.

(41) *exactly n* $\quad \exists^{x=n}[\phi]\ (\psi) := \mathbf{M}([x] \wedge \phi \wedge \psi) \wedge {}^{x=n}$

That is, numeral modifiers *exactly*, *at most*, *at least* etc. can be thought of as functions that take a bare numeral as their argument and introduce:

- a maximization operator $\mathbf{M}$ that scopes over the random assignment and the restrictor and nuclear scope formulas
- a post-supposition that consists of the cardinality requirement ordinarily contributed by the bare numeral

The translation of sentence (1) derives the intuitively-correct cumulative truth conditions.

(42) a. $\exists^{x=3}[\textsc{boy}(x)]\ (\exists^{y=5}[\textsc{movie}(y)]\ (\textsc{see}(x, y)))$
b. $\mathbf{M}([x] \wedge \textsc{boy}(x) \wedge \mathbf{M}([y] \wedge \textsc{movie}(y) \wedge \textsc{see}(x, y)) \wedge {}^{y=5}) \wedge {}^{x=3}$
c. $\mathbf{M}([x] \wedge \textsc{boy}(x) \wedge \mathbf{M}([y] \wedge \textsc{movie}(y) \wedge \textsc{see}(x, y))) \wedge {}^{y=5} \wedge {}^{x=3}$
d. $\mathbf{M}([x] \wedge \textsc{boy}(x) \wedge [y] \wedge \textsc{movie}(y) \wedge \textsc{see}(x, y)) \wedge y = 5 \wedge x = 3$

All the formulas in (42) are truth-conditionally equivalent (given the definition of truth in (40)):

- (42a) is the formula we derive following our compositional translation schemas
- (42b) unpacks (42a) based on the abbreviations defined for each of the translation schemas
- (42c) is just like (42b) except that the post-supposition ${}^{y=5}$ contributed by the direct object is extracted from the scope of the $\mathbf{M}$ operator contributed by the subject
  - the two formulas are equivalent because post-suppositions are simply collected and passed on from local context to local context and are required to be satisfied only relative to the final output context
- (42d) is just like (42c) except that the update-final post-suppositions ${}^{y=5}$ and ${}^{x=3}$ are converted into at-issue tests $y = 5$ and $x = 3$
  - the truth-conditional equivalence of the two formulas follows from the definition of truth and the semantics of post-suppositions and at-issue tests

Just as before, if we elaborate on sentence (1) with *They$_x$ liked them$_y$*, we derive the intuitively-correct interpretation for the entire discourse:[4]

- every one of the three boys liked every movie he saw (and not the movies some other boy saw)

The proposed analysis of modified numerals makes use of three crucial ingredients:

- evaluation pluralities (i.e., sets of assignments)
- maximization operators over such evaluation pluralities
- post-suppositions and their unusual 'scoping' behavior

The goal of the following three sections is to provide independent evidence for each of these ingredients, in turn.

[4] We can capture the fact that *at most n* numerals do not have an existential entailment – unless they are anaphorically retrieved later on – in various ways. One technical solution (based on Brasoveanu 2009) that would enable us to still work with total assignments involves: (*i*) adding a dummy individual # that is a universal falsifier, (*ii*) letting random assignments $[x]$, $[y]$ etc. introduce this individual as well as regular individuals and (*iii*) systematically discarding it when we evaluate the cardinality of the sets stored by variables x, y etc.

3 Universal Quantifiers

Evaluation pluralities enable us to capture cumulative readings for bare and modified numerals – and also for distributive universal quantifiers like *every* (see Schein 1993, Kratzer 2000, Champollion 2009).[5]

Consider the sentence below:

(43) Three^x copy editors (between them) caught every^y mistake in the manuscript.

- cumulative reading:
  - there are three copy editors such that each of them caught at least one mistake
  - every mistake was caught by at least one of the three editors

Distributive universal quantification is translated as follows:

(44) $\forall x[\phi]\ \delta(\psi) := \mathbf{M}x(\phi) \wedge \delta(\psi)$

- $\mathbf{M}x(\phi)$: we introduce the set of all individuals x that satisfy the restrictor ϕ
- $\delta(\psi)$: we then check that *each* of these individuals also satisfies the nuclear scope ψ

The maximization operator $\mathbf{M}x$ is the selective counterpart of the unselective/adverbial maximization operator $\mathbf{M}$ ('selective' and 'unselective' in the sense of Lewis 1975).

(45) $[\![\mathbf{M}x(\phi)]\!]^{\langle G[\zeta],H[\zeta']\rangle} = \mathbb{T}$ iff $[\![[x] \wedge \phi]\!]^{\langle G[\zeta],H[\zeta']\rangle} = \mathbb{T}$ and there is no H' s.t. $H(x) \subsetneq H'(x)$ and $[\![[x] \wedge \phi]\!]^{\langle G[\zeta],H'[\zeta']\rangle} = \mathbb{T}$.

- unselective $\mathbf{M}$ maximizes over sets of assignments H
- selective $\mathbf{M}x$ maximizes over sets of individuals $H(x)$

Using an unselective maximization operator for modified numerals is justified by the fact that their modifier can be non-adjacent/adverbial:

(46) Three boys saw five movies, exactly/precisely/at (the) most.

(47) The league limits teams to playing two games in a row – or, at the most, four games in five days, NBA spokesman Tim Frank says. (COCA)

[5] I am indebted to Lucas Champollion for many helpful comments that greatly improved this section.

- note that the cumulative reading of (47) is one in which *at the most* simultaneously targets *four games* and *five days*

(48) It was a kind of pension where, at the most, there were four or five guests. (COCA)

(49) The reproductive ratio, the spread from one person to another, is no more than two, which means that
One person, at most, infects two others . . . (COCA)

We now need to define the distributivity operator δ. Let us first ignore post-suppositions and define the basic notion of distributivity over sets of assignments (based on Brasoveanu 2008):

(50) $[\![\delta(\phi)]\!]^{\langle G,H\rangle} = \mathbb{T}$ iff there exists a relation $\mathcal{R}$ between assignments and sets of assignments, i.e., of the form $\mathcal{R}(g, K)$, s.t.
a. $G = \mathbf{Dom}(\mathcal{R})$ and $H = \bigcup \mathbf{Ran}(\mathcal{R})$
b. for all g and all K, if $\mathcal{R}(g, K)$ holds, then $[\![\phi]\!]^{\langle \{g\},K\rangle} = \mathbb{T}$

- we distributively update a set of assignments G with a formula ϕ by updating each singleton subset $\{g\} \subseteq G$ with ϕ and taking the union of the resulting output sets of assignments K

Distributivity with post-suppositions: we need the distributivity operator δ to discharge any of the post-suppositions contributed by the formula in its scope.

- unlike regular formulas or the maximization operator $\mathbf{M}$, distributivity operators do not merely inherit the input quantificational alternatives and elaborate on them
  - where a quantificational alternative is a set of assignments
- distributivity breaks each input quantificational alternative, i.e., each input set of assignments G, into singleton subsets $\{g\}$ and locally generates a new set of quantificational alternatives K
- this is the essence of semantic scope/co-variation in our system
- post-suppositions globally constrain the set of quantificational alternatives *relative to which they have been introduced*
  - they should 'outscope' operators that do not introduce new quantificational alternatives like $\mathbf{M}$
  - but they should not 'outscope' distributivity operators
- in a sense, distributivity operators are to quantificational alternatives what clausal boundaries are to movement in syntax: they mark locality domains/barriers

(51) $[\![\delta(\phi)]\!]^{\langle G[\zeta], H[\zeta']\rangle} = \mathbb{T}$ iff $\zeta = \zeta'$ and there exists a relation $\mathcal{R}$ between assignments and sets of assignments, i.e., of the form $\mathcal{R}(g, K)$, s.t.

a. $G = \mathbf{Dom}(\mathcal{R})$ and $H = \bigcup \mathbf{Ran}(\mathcal{R})$

b. for all g and all K such that $\mathcal{R}(g, K)$, there is a (possibly empty) set of formulas $\{\psi_1, \ldots, \psi_m\}$ s.t. $[\![\phi]\!]^{\langle \{g\}[\zeta], K[\zeta \cup \{\psi_1, \ldots, \psi_m\}]\rangle} = \mathbb{T}$ and $[\![\psi_1 \wedge \ldots \wedge \psi_m]\!]^{\langle K[\zeta], K[\zeta]\rangle} = \mathbb{T}$

Thus, just like presuppositions – or scalar implicatures in theories like Chierchia et al (2009), post-suppositions are not always satisfied globally, but can be satisfied/discharged at intermediate points in the semantic composition, i.e., in more local output contexts.

Given that we work with quantificational alternatives, and not with focus alternatives, as in Krifka (1999), we expect various *quantificational* operators (universals, modals, attitude verbs etc.) to block the 'projection' of post-suppositions and discharge them locally, in their scope.

The translation of sentence (43) is provided below:

(52) a. $\exists x[x = 3 \wedge \textsc{editor}(x)]\ (\forall y[\textsc{mistake}(y)]\ \delta(\textsc{catch}(x, y)))$

b. $[x] \wedge x = 3 \wedge \textsc{editor}(x) \wedge \mathbf{M}y(\textsc{mistake}(y)) \wedge \delta(\textsc{catch}(x, y))$

c. $[x] \wedge x = 3 \wedge \textsc{editor}(x) \wedge \mathbf{M}y(\textsc{mistake}(y)) \wedge \textsc{catch}(x, y)$

- we introduce a set x of three editors and the set y of all mistakes and check that, for every assignment h in the resulting output state H, the editor $h(x)$ caught the mistake $h(y)$
- except for the presence of the operator $\mathbf{M}x$ introduced by universals, the cumulative representation in (52c) is parallel to one with two bare numerals in (32b)

Had we used unselective maximization $\mathbf{M}([y] \wedge \textsc{mistake}(y))$, we would have introduced all the mistakes relative to each one of the three editors and we would have incorrectly required each editor to catch every mistake, i.e., we would have failed to capture the cumulative reading.

- we will use unselective maximization for necessity modals, which do not exhibit cumulative readings

The distributivity operator δ is semantically vacuous in (52), so it can be omitted (as we did in (52c)). But it isn't always vacuous.

Consider the example below (from Kratzer 2000):

(53) Every^x copy editor caught 500^y mistakes in the manuscript.

- as Kratzer (2000) notes, this sentence does not have a cumulative reading to the effect that, between them, the copy editors caught a total of 500 mistakes in the manuscript
- the only available reading is the distributive one: every copy editor is such that s/he caught 500 mistakes

We derive the distributive reading if the universal quantifier takes scope over the numeral. That is, cumulative readings are possible with universal quantifiers only if they have narrow scope relative to the numerals they 'cumulate' with.

- as long as the non-surface scope *500 >> every* is somehow blocked for sentence (53), we correctly derive the unavailability of the cumulative reading

The translation of the surface-scope reading *every >> 500* for sentence (53) is provided below:

(54) a. $\forall x[\textsc{editor}(x)]\ \delta(\exists y[y = 500 \wedge \textsc{mistake}(y)]\ (\textsc{catch}(x, y)))$

b. $\mathbf{M}x(\textsc{editor}(x)) \wedge \delta([y] \wedge y = 500 \wedge \textsc{mistake}(y) \wedge \textsc{catch}(x, y))$

- we introduce the set of all copy editors x and we check that *each* of them caught 500 mistakes
- the distributivity operator δ is not semantically vacuous, so it cannot be omitted

This analysis also generalizes to the mixed cumulative-distributive sentence below (from Schein 1993):

(55) Three^x video games taught every^y quarterback two^z new plays.

As Kratzer (2000) observes:

- *every^y quarterback* and *three^x video games* are related cumulatively: between them, a total of three video games taught all the quarterbacks
- but *every^y quarterback* behaves just like an ordinary distributive quantifier with respect to *two^z new plays*: every quarterback learned two possibly different plays

In our framework, this follows if we preserve the surface-scope relations between the three quantifiers: *three >> every >> two.*

The resulting translation, which derives the intuitively-correct truth conditions, is provided below:

(56) a. $\exists x[x = 3 \wedge \textsc{game}(x)]\ (\forall y[\textsc{q.back}(y)]\ \delta(\exists z[z = 2 \wedge \textsc{play}(z)]\ (\textsc{teach}(x, y, z))))$

b. $[x] \wedge x = 3 \wedge \text{GAME}(x) \wedge \mathbf{M}y(\text{Q.BACK}(y)) \wedge \delta([z] \wedge z = 2 \wedge \text{PLAY}(z) \wedge \text{TEACH}(x, y, z))$

In all the above examples, we set the granularity of the distributivity contributed by *every* to atoms – but we don't have to.

- we assumed for simplicity that our domain of individuals contains only atoms, but it may very well contain non-atomic individuals
- the use of evaluation pluralities (sets of assignments) does not entail that we should not have domain-level pluralities (contra van den Berg 1996 among others)
- the same applies to other domains: we can and should allow for time intervals (convex sets of instants, which are plural), intervals over degrees, mass nouns and measured portions thereof, (sets of) locations etc.

The need for this variety of domain pluralities is exemplified by (57) below (from Schwarzschild 1996), which universally quantifies over non-atomic individuals that contain three atoms, and the naturally-occurring examples in (58) through (61) from COCA that quantify over intervals:

(57) I observed that every three houses {formed a block/were built in the same style}.

(58) I'm going to play two more years at the most.

(59) [Years back when I was a kid, I used to see eight-foot-thick ice, 14-feet-thick ice.]
Forty years later, I'm now seeing ice, at the most, at less than two feet thick.

(60) They reported that the projects reached, at the most, less than one percent of the United States high school population.

(61) The composite indicator endeavor, at the most, would cost five to ten million dollars.

Finally, we capture the distributive reading of (1) by means of the operator δ. Distributive modified numerals have a δ operator over their nuclear scope.

(62) $\exists^{x=n}[\phi]\ \delta(\psi) := \mathbf{M}([x] \wedge \phi \wedge \delta(\psi)) \wedge\ ^{x=n}$

(63) a. $\exists^{x=3}[\text{BOY}(x)]\ \delta(\exists^{y=5}[\text{MOVIE}(y)]\ \delta(\text{SEE}(x, y)))$

b. $\mathbf{M}([x] \wedge \text{BOY}(x) \wedge \delta(\mathbf{M}([y] \wedge \text{MOVIE}(y) \wedge \text{SEE}(x, y)) \wedge\ ^{y=5})) \wedge\ ^{x=3}$

c. $\mathbf{M}([x] \wedge \text{BOY}(x) \wedge \delta(\mathbf{M}([y] \wedge \text{MOVIE}(y) \wedge \text{SEE}(x, y)) \wedge y = 5)) \wedge x = 3$

4 Implicatures

This section provides independent evidence for analyzing modified numerals by means of a maximization operator over evaluation pluralities.

4.1 Scalar Implicatures and Referential Uncertainty

Basic idea: modified numerals do not trigger scalar implicatures – unlike bare numerals/indefinites – because they contribute a maximization operator **M** that effectively eliminates referential uncertainty.

In any given world, the variable introduced by a modified numeral can be associated with only one set of values: the set of all entities that satisfy the restrictor and nuclear scope of the modified numeral. This is shown by the contrast below (from Umbach 2006; see also Szabolcsi 1997, Swart 1999 and Krifka 1999):

(64) a. { Two / #At least two } boys were selling coke.

b. They were wearing black leather jackets.

c. Perhaps there were others also selling coke, but I didn't notice.

- if there were more than two boys selling coke, the variable introduced by the bare numeral *two* can take different sets of two boys as values
- in the present system, this referential indeterminacy/uncertainty is captured by the fact that the output contexts, i.e., matrices, obtained after the update with a bare numeral might assign different sets of values to the variable contributed by the bare numeral
- in contrast, the variable introduced by *at least two* has only one possible value: the set of all boys who were selling coke
- this kind of determined reference is captured by the fact that, in any given world, all output contexts obtained after the update with a modified numeral assign the same value to variable contributed by the modified numeral

Thus, the proposal is that scalar implicatures are made possible by items that allow for referential indeterminacy/uncertainty. It is this semantic uncertainty that kicks off the pragmatic inferential process whose result is the addition of scalar implicatures.
However, note that the *perhaps* continuation is felicitous in many examples from COCA:

(65) In February, the French government-controlled engineering giant Areva, the world's biggest nuclear power plant construction company, announced it would build at least two, and perhaps six, EPR nuclear reactors in India.

(66) An American platoon surprised an armed Taliban column on a forested ridgeline at night, and killed at least 13 insurgents, and perhaps many more, with rifles, machine guns, Claymore mines, hand grenades and a knife.

(67) Against the Khwrazmian Empire, Chinggis Khan used at least four and perhaps five routes.

(68) At least one and perhaps two of the first four Rotarians were Masons.

(69) Up until the last 20 years, vaccines contained at least 200 and perhaps more than 3,000 antigens.

(70) In two days of mayhem, U.S. soldiers killed at least 10 and perhaps as many as 17 people, according to military and hospital officials.

(71) More recent estimates indicate that militia groups are active in at least 35 states and perhaps in all 50 by now.

(72) So far, at least 40, perhaps as many as 52 are known dead, including 12 children.

(73) At least 4,000 people, and perhaps as many as 6,500, were killed.

But here are some examples from COCA in which plural anaphora does seem to be maximal:

(74) *At least*:

a. There were at least 40 shots. They were single shots but fairly close together.

b. Most Europeans speak at least two languages and they speak them well by the time they're out of school.

c. In two sorties, jets fired at least 11 heavy-detonation projectiles. They lit up the night sky.

d. The FBI, according to reports, believes at least four men were involved in the bombing, perhaps more, some of them with ties to the most radical of the right-wing Militia groups.

e. [Typical non-maximal example – emphasizes epistemic uncertainty] The literature reveals at least two forms of sexism. They are (1) traditional sexism [...] and (2) modern sexism [...].

(75) *At most*:

a. Fighting was scarce and there was a very small number of guerrillas – in 1978, today's main organization, the FARC, had at most 500 soldiers – and they prowled the most isolated areas of the country.

(76) *Up to*:

a. Program up to 37 alerts; they'll reset automatically at midnight.

b. So you can submit up to three entries. They'll be judged on meaning, naturalness of syntax, originality and overall elegance.

c. The party, held in the courtyard every Tuesday and Thursday from April to October, attracts up to 80 people. And they all have one thing in common: their dogs.

d. The United States is prepared to send up to 30,000 troops to Somalia. They would join a U.N. peacekeeping and food distribution effort.

e. Iraq is said to be holding up to 30,000 Kuwaiti prisoners. They were taken during the occupation of Kuwait.

(77) *As many as*:

a. Out of some 14,000 wildebeests, as many as 3,000 behaved as permanent residents. They could be found in specific areas.

b. There were about 50 or a hundred black bears in New Jersey in 1970; now there may be as many as 3,000. They've been causing a lot of problem in suburban areas.

c. There are, depending on how you count them, perhaps as many as 720 national laboratories. They have collective budgets of more than twenty billion dollars.

d. Between shareware and commercial products, there may be as many as 50 of these utilities, and they vary in features and functionality.

e. Huge screens track as many as 200 airplanes at a time as they arc over the Atlantic Ocean.

4.2 Epistemic Uncertainty and Modal Readings

But: referential certainty is distinct from epistemic certainty.

- imagine that our contexts are not simply sets of assignments G, H etc. but, in the spirit of Heim (1982), they are pairs of a world and a set of assignments $\langle w, G\rangle$, $\langle w', H\rangle$ etc.
- at any point in discourse, the *information state* at that point consists of all the $\langle$*world*, *set-of-assignments*$\rangle$ pairs that are still live options, i.e., that are compatible with all the previous updates
- the referential uncertainty of the participants in the conversation is encoded by the second member of all such pairs: all the sets of assignments in the current information state
- the epistemic uncertainty is encoded by the first member of all the pairs: all the worlds in the current information state; this set of worlds is the current Context Set (Stalnaker 1978)

Modified numerals are referentially determined, but epistemically uncertain:

- if we fix the world, the variable contributed by the modified numeral has only one value – the supremum in the set-inclusion partial order[6]

[6]Assuming the global supremum exists, which usually is the case. Otherwise, given the definition of the **M** operator, the modified numeral non-deterministically introduces a local maximum.

- but this supremum set may vary from world to world

The pragmatics of modified numerals:

- we can use a modified numeral – as opposed to its bare counterpart – only if we are epistemically uncertain about the cardinality of the maximal set of entities introduced by the numeral

Whenever this epistemic uncertainty requirement is not satisfied, modified numerals are infelicitous – in contrast to comparative quantifiers, as observed in Nouwen (2009) (see also references therein).

That is, modified numerals trigger epistemic implicatures of the kind proposed in Büring (2008) for *at least*.

(78) Paul has at least four guitars.

This non-modalized sentence is cooperatively used only if the speaker:

- is certain that Paul has four guitars
- considers it possible that Paul has exactly four guitars
- considers it possible that Paul has more than four guitars

Consider now the example below (from Nouwen 2009; see also Geurts & Nouwen 2007 and Krifka 2007):

(79) Jasper invited maximally 50 people to his party.

Extending the proposal in Büring (2008), this sentence should be cooperatively used only if the speaker:

- is certain that Jasper did not invite more than 50 people to his party
- considers it possible that Jasper invited exactly 50 people to his party
- considers it possible that Jasper invited less than 50 people to his party

That is, (79) is normally taken to indicate that the speaker does not know exactly how many people Jasper invited. So:

- (79) is interpreted as being about the non-trivial (i.e., non-singleton) *range* of cardinalities possible at that point in discourse
- it is therefore unacceptable for a speaker to utter (79) and continue with: *43, to be precise*
- we don't need to derive the modal readings of indicative sentences with modified numerals by inserting covert modals or speech act operators (contra Geurts & Nouwen 2007, Krifka 2007, Nouwen 2009)

The same pragmatic infelicity due to the epistemic uncertainty requirement can arise intra-sententially: if we know what the word *hexagon* means, the sentences below (from Nouwen 2009) with modified numerals are infelicitous (but not those with comparative quantifiers):

(80) a. A hexagon has fewer than 11 sides.
b. #A hexagon has at most / maximally / up to 11 sides.

(81) a. A hexagon has more than 3 sides.
b. #A hexagon has at least / minimally 11 sides.

5 Modals and Modified Numerals

This section provides independent evidence for the analysis of modified numerals in terms of post-suppositions, which exhibit an unusual 'scoping' behavior.

To analyze modal verbs, we expand our language and its models in the usual way:

- we add a set of possible worlds $\mathfrak{W}$ disjoint from $\mathfrak{D}$ and variables over possible worlds $w^*, w, w', w_1, w_2, \ldots$
- we relativize the basic interpretation function $\mathfrak{I}$ to worlds: for any world $\mathfrak{u}$ and any n-ary relation R, $\mathfrak{I}_\mathfrak{u}(R) \subseteq \mathfrak{D}^n$
- we take the variable w^* to be the designated variable for the actual world: for any discourse-initial set of assignments G that we will consider, we assume that $|G(w^*)| = 1$
- w^* performs the same function as the world w in the Heim (1982)-style contexts introduced above, i.e., contexts of the form $\langle w, G\rangle$ that pair a set of assignments G with a world w
- lexical relations are interpreted in the expected way

(82) $[\![R_{w^*}(x_1, \ldots, x_n)]\!]^{\langle G[\zeta], H[\zeta']\rangle} = \mathbb{T}$ iff $G = H$, $\zeta = \zeta'$ and for all $h \in H$, $\langle h(x_1), \ldots, h(x_n)\rangle \in \mathfrak{I}_{h(w^*)}(R)$.

- cardinality requirements are also relativized to possible worlds

(83) $G_{w=\mathfrak{u}} := \{g \in G : g(w) = \mathfrak{u}\}$

(84) $[\![x =_w n]\!]^{\langle G[\zeta],H[\zeta']\rangle} = \mathbb{T}$ iff $G = H$, $\zeta = \zeta'$ and for each $\mathfrak{u} \in H(w)$, $|H_{w=\mathfrak{u}}(x)| = n$

– for each world $\mathfrak{u}$ that is one of the values of w, the cardinality of x relative to $\mathfrak{u}$ is n

(85) $[\![x \leq_w n]\!]^{\langle G[\zeta],H[\zeta']\rangle} = \mathbb{T}$ iff $G = H$, $\zeta = \zeta'$ and for each $\mathfrak{u} \in H(w)$, $|H_{w=\mathfrak{u}}(x)| \leq n$

– for each world $\mathfrak{u}$ that is one of the values of w, the cardinality of x relative to $\mathfrak{u}$ is less than or equal to n

5.1 Minimal Requirements

Consider now the sentence below (based on Nouwen 2009):

(86) Jasper^x should^w read at least ten^y books (to please his$_x$ mother).

- the most salient reading of (86): the minimum number of books that Jasper is *allowed* to read (if he wants to please his mother) is 10
- as Nouwen (2009) observes, under standard assumptions about the semantics of minimizers and necessity modals, there is no satisfactory analysis of minimal requirements

The intuitively-correct truth conditions follow automatically in the present framework.

We analyze necessity modals as distributive universal quantifiers in the modal domain:

- except that they introduce a non-selective maximization operator $\mathbf{M}([w] \wedge \ldots)$ instead of a selective operator $\mathbf{M}w(\ldots)$
- unselective maximization captures the fact that, unlike universal quantifiers over individuals, necessity modals do not have cumulative readings
- unselective maximization can also be seen as the a reflection of the more 'adverbial' nature of modals

(87) $\mathbf{NEC}w(\phi) := \mathbf{M}([w] \wedge R_{w^*}(w)) \wedge \delta(\phi)$

- R is a contextually-provided accessibility relation (modal base +/− built-in ordering source)
- $R_{w^*}(w)$ is intuitively interpreted as: w is an R-accessible world from w^*

Sentence (86) is translated as:

(88) a. $\mathbf{NEC}w(\exists x[x = \text{JASPER}]\ (\exists^{y \geq_w 10}[\text{BOOK}_w(y)]\ (\text{READ}_w(x,y))))$
b. $\mathbf{M}([w] \wedge R_{w^*}(w)) \wedge$
$\delta([x] \wedge x = \text{JASPER} \wedge \mathbf{M}([y] \wedge \text{BOOK}_w(y) \wedge \text{READ}_w(x,y)) \wedge {}^{y \geq_w 10})$
c. $\mathbf{M}([w] \wedge R_{w^*}(w)) \wedge$
$\delta([x] \wedge x = \text{JASPER} \wedge \mathbf{M}([y] \wedge \text{BOOK}_w(y) \wedge \text{READ}_w(x,y)) \wedge y \geq_w 10)$

The update in (88) instructs us to:

- introduce all the worlds w that are R-accessible from the actual world w^*; these are the deontically-ideal worlds that the modal verb *should* universally quantifies over
- distributively check that, for each ideal world w: if we store Jasper in x and in y all the books that Jasper read in world w, the cardinality of the set of books is at least 10

That is:

- Jasper reads at least 10 books in every deontically-ideal world w
- so the minimum number of books that Jasper is allowed to read is 10

5.2 Maximal Permissions

We correctly analyze maximal permissions, e.g., (89) below (from Nouwen 2009):

(89) Jasper^x is allowed^w to invite at most ten^y people.

- the most salient reading of (89) is: the maximum number of people Jasper is allowed to invite is 10

We take possibility modals to be the modal counterpart of maximal *some*. We analyze *some* and *might* / *allow* etc. in a way that is parallel to modified numerals:

- a maximization operator $\mathbf{M}$ followed by a post-suppositional cardinality requirement

The maximization operator $\mathbf{M}$ is justified by the maximal (E-type) anaphora exemplified by the following well-known examples (see Evans 1977, 1980 and Roberts 1987, 1989):

(90) Harry bought some sheep. Bill vaccinated them.

- the most salient reading: Bill vaccinated *all* the sheep that Harry bought

(91) A wolf might come in. It would eat Jasper first.

- the most salient reading: for *any* epistemically-possible scenario of a wolf coming in, the wolf eats Jasper first

(92) *some* (extensional version) $\exists^{x>1}[\phi]\ (\psi) := \mathbf{M}([x] \wedge \phi \wedge \psi) \wedge\ ^{x>1}$

(93) $\mathbf{POS}w(\phi) := \exists^{w>1}[R_{w^*}(w)]\ (\phi) = \mathbf{M}([w] \wedge R_{w^*}(w) \wedge \phi) \wedge\ ^{w>1}$

Sentence (89) is translated as:

(94) a. $\mathbf{POS}w(\exists x[x = \text{JASPER}]\ (\exists^{y\leq_w 10}[\text{PERSON}_w(y)]\ (\text{INVITE}_w(x,y))))$
b. $\mathbf{M}([w] \wedge R_{w^*}(w) \wedge [x] \wedge x = \text{JASPER} \wedge \mathbf{M}([y] \wedge \text{PERSON}_w(y) \wedge \text{INVITE}_w(x,y)) \wedge\ ^{y\leq_w 10}) \wedge\ ^{w>1}$
c. $\mathbf{M}([w] \wedge R_{w^*}(w) \wedge [x] \wedge x = \text{JASPER} \wedge \mathbf{M}([y] \wedge \text{PERSON}_w(y) \wedge \text{INVITE}_w(x,y))) \wedge\ ^{y\leq_w 10} \wedge\ ^{w>1}$
d. $\mathbf{M}([w] \wedge R_{w^*}(w) \wedge [x] \wedge x = \text{JASPER} \wedge [y] \wedge \text{PERSON}_w(y) \wedge \text{INVITE}_w(x,y)) \wedge y \leq_w 10 \wedge w > 1$

The update in (94) instructs us to:

- introduce all the worlds w that are R-accessible from the actual world w^* (i.e., deontically-ideal) such that Jasper invites some people in w
- for each such world w, store in y all the people invited by Jasper
- finally, check that there is more than 1 such ideal world w and that the cardinality of the set y of invited people in *each* world w taken individually is at most 10

5.3 Distributive Permissions

Analyzing possibility modals in parallel to modified numerals predicts that they can also have distributive readings of the following form:

(95) $\mathbf{POS}w(\delta(\phi)) := \exists^{w>1}[R_{w^*}(w)]\ \delta(\phi) = \mathbf{M}([w] \wedge R_{w^*}(w) \wedge \delta(\phi)) \wedge\ ^{w>1}$

The resulting distributive translation of sentence (89) is:

(96) a. $\mathbf{POS}w(\delta(\exists x[x = \text{JASPER}]\ (\exists^{y\leq_w 10}[\text{PERSON}_w(y)]\ (\text{INVITE}_w(x,y)))))$
b. $\mathbf{M}([w] \wedge R_{w^*}(w) \wedge \delta([x] \wedge x = \text{JASPER} \wedge \mathbf{M}([y] \wedge \text{PERSON}_w(y) \wedge \text{INVITE}_w(x,y)) \wedge\ ^{y\leq_w 10})) \wedge\ ^{w>1}$
c. $\mathbf{M}([w] \wedge R_{w^*}(w) \wedge \delta([x] \wedge x = \text{JASPER} \wedge \mathbf{M}([y] \wedge \text{PERSON}_w(y) \wedge \text{INVITE}_w(x,y)) \wedge y \leq_w 10)) \wedge w > 1$

The update in (96) is interpreted as:

- there is more than one world w that is R-accessible from the actual world w^* such that the maximum number of people Jasper invites in w is at most 10
- that is, inviting at most 10 people is something that Jasper is allowed to do

This rather weak reading is not intuitively available for sentence (89). I follow Nouwen (2009) (see, e.g., p. 17) and assume that such readings are blocked by the availability of (and competition with) the parallel construction with a bare numeral instead of a modified numeral.

In general, the proposal that bare numerals can block modified numerals predicts that whenever

(*i*) an operator, e.g., $\mathbf{POS}w$, can have both a cumulative and a distributive reading

(*ii*) this operator has a modified numeral in its scope

the distributive reading that locally discharges the post-supposition contributed by the modified numeral competes with – and is blocked by – the parallel construction with a bare numeral instead of the modified numeral, which has no post-supposition.

For example, the bare numeral counterpart of sentence (89) is:

(97) Jasper^x is allowed^w to invite ten^y people.

The possibility modal can be interpreted cumulatively or distributively:

(98) a. $\mathbf{POS}w(\exists x[x = \text{JASPER}]\ (\exists y[y =_w 10 \wedge \text{PERSON}_w(y)]\ (\text{INVITE}_w(x,y))))$
b. $\mathbf{M}([w] \wedge R_{w^*}(w) \wedge [x] \wedge x = \text{JASPER} \wedge [y] \wedge y =_w 10 \wedge \text{PERSON}_w(y) \wedge \text{INVITE}_w(x,y)) \wedge w > 1$

(99) a. $\mathbf{POS}w(\delta(\exists x[x = \text{JASPER}]\ (\exists y[y =_w 10 \wedge \text{PERSON}_w(y)]\ (\text{INVITE}_w(x,y)))))$
b. $\mathbf{M}([w] \wedge R_{w^*}(w) \wedge \delta([x] \wedge x = \text{JASPER} \wedge [y] \wedge y =_w 10 \wedge \text{PERSON}_w(y) \wedge \text{INVITE}_w(x,y))) \wedge w > 1$

Either way, we obtain the same reading:

- there is more than one world w that is R-accessible from the actual world w^* such that Jasper invites 10 people in w
- that is, inviting 10 people (or less) is something that Jasper is allowed to do

We can use the same blocking mechanism to derive the infelicity of minimal permissions like sentence (100) below.

(100) #A course is allowed to have at least four registered students (to be approved by the administration).

(101) A course is allowed to have four registered students.

(102) A course must have at least four registered students (to be approved by the administration).

The distributive reading of the possibility modal in (100) is blocked by the bare numeral construction in (101), just as before.

The cumulative reading of the possibility modal in (100) is presumably blocked by the alternative, unambiguous construction in (102), where the possibility modal is replaced with its unambiguously-distributive universal counterpart.

Open question: why is that the cumulative reading of the possibility sentence in (100) is blocked by the necessity sentence in (102), but the cumulative reading of (89) above seems to not be blocked by its necessity counterpart in (103) below:

(103) Jasper^x is required^w to invite at most ten^y people.

6 Conclusion

- we introduced a framework in which we distinguished between two kinds of pluralities:
  - evaluation plurality, i.e., sets of assignments
  - domain plurality, i.e., non-atomic individuals – which we have not discussed, but which we would allow (contra van den Berg 1996 among others)
- the maximization operator **M** and the distributivity operator δ are to evaluation pluralities what the familiar Link-style sum and distributivity operators are to domain pluralities
- cumulativity is just non-distributivity with respect to evaluation pluralities, while collectivity (group readings, 'partial covers' etc.) is just non-distributivity with respect to domain pluralities
- post-suppositions are constraints on output contexts – in contrast to presuppositions, which constrain input contexts
- just as presuppositions – or implicatures in theories like Chierchia et al (2009), post-suppositions can be satisfied / discharged non-globally, e.g., in the scope of distributivity operators
- unlike presuppositions, post-suppositions are part of the proposal to update the Common Ground (since they are part and parcel of the regular truth-conditions), hence they can be challenged, questioned etc.
- post-suppositions are distinct from regular at-issue meaning with respect to their evaluation / update order: they can constrain the final, global output context
- post-suppositions constrain quantificational, not focus (Krifka 1999), alternatives (where a quantificational alternative is a possible output set of assignments) – so, it is expected that various *quantificational* operators (universals, modals, attitude verbs etc.) block the 'projection' of post-suppositions and discharge them locally, in their scope
- this approach does not apply to modified numerals across the board, but only to what Nouwen (2009) identifies as type B modified numerals
  - type A modified numerals, e.g., comparative quantifiers like *more than n*, *fewer than n* etc., should probably be analyzed as in Nouwen (2009), i.e., in terms of degree quantification along the lines of Hackl (2000)
- enriching contexts of evaluation with post-suppositions follows the same basic strategy / insight as classical dynamic semantics:
  - enriching contexts of evaluation (and, therefore, the inventory of operators that can be defined over them) enables us to keep our interpretation compositional and surface-based
  - the reason: local operations over contexts can have global effects because the recursive definition of truth and satisfaction preserves and passes on these local contextual changes
- the post-suppositional account of modified numerals is not a theory of scalar implicatures: cardinality post-suppositions (in combination with maximization operators) encode scalar meaning and this meaning is separate from regular at-issue meaning, but it is part of the grammar of overt lexical items, namely numeral modifiers
  - however, there are interesting parallels between the post-suppositional account of modified numerals in terms of quantificational alternatives and accounts of scalar implicatures in terms of focus alternatives like Chierchia et al (2009) and references therein, which take scalar implicatures to also be part of the grammar

Future directions:

- adverbial / 'floated' uses of *exactly* / *precisely* / *maximally* / *approximately* etc.[7]

[7] I am indebted to Pranav Anand, Jim McCloskey and an anonymous Amsterdam Colloquium 2009 reviewer for discussion of this point.

(104) Three boys exactly saw (exactly / precisely) five movies.

(105) *Three boys exactly saw five movies exactly / precisely.

- the operator **M** maximizes over cases / assignments, so this kind an examples does not seem completely out of reach
- the analysis of adverbial *exactly* / *precisely* etc. may be parallel to the analysis of adverbs of quantification like *always*, *usually* etc.

• the availability of cumulative vs distributive readings is sensitive to questions under discussion:[8] cumulative readings seem unavailable as answers to single *who* / *how many* questions, but available as answers to multiple *wh*-questions; moreover, sentences with such cumulative readings seem to have a particular intonation pattern

(106) How many boys saw exactly five movies?

(107) How many boys saw how many movies?

• cumulative readings are also unavailable in cases in which a parallelism discourse relation needs to be established, e.g.:

(108) Mary saw exactly five movies and exactly three boys did too / saw exactly five movies too.

• this is not unexpected: quantifier scope, which also involves manipulating the evaluation order of certain expressions, is sensitive to questions under discussion and discourse relations

The broader question is: how do focus alternatives and quantificational alternatives interact?

This interaction can already be observed in simple *wh*-questions, as van Rooy (1998) notices:

(109) Who^x went to the party and what did they$_x$ bring as a present?

(110) Q: Who^x went to the party?
A: I don't know, but Jasper wasn't one of them$_x$.

(111) Which^x guest brought which^y present and where did they$_x$ buy them$_y$?

[8] I am indebted to an anonymous Amsterdam Colloquium 2009 reviewer for bringing this point to my attention.

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A DPlL with Post-suppositions (DPlL^{post})

The following is the definition of the DPlL^{post} (extensional) interpretation function $[\![\cdot]\!]^{\langle G[\zeta],H[\zeta']\rangle}$, where ζ and ζ' are two sets of formulas.

(1) $H[x]G := \begin{cases} \text{for all } h \in H, \text{ there is a } g \in G \text{ such that } h[x]g \\ \text{for all } g \in G, \text{ there is a } h \in H \text{ such that } h[x]g \end{cases}$

(2) $G(x) := \{g(x) : g \in G\}$

(3) $|G(x)|$ is the cardinality of the set of individuals $G(x)$

(4) $[\![R(x_1, \ldots, x_n)]\!]^{\langle G[\zeta],H[\zeta']\rangle} = \mathbb{T}$ iff $G = H$, $\zeta = \zeta'$ and $\langle h(x_1), \ldots, h(x_n)\rangle \in \mathfrak{I}(R)$, for all $h \in H$

(5) $[\![x = n]\!]^{\langle G[\zeta],H[\zeta']\rangle} = \mathbb{T}$ iff $G = H$, $\zeta = \zeta'$ and $|H(x)| = n$

(6) $[\![x \leq n]\!]^{\langle G[\zeta],H[\zeta']\rangle} = \mathbb{T}$ iff $G = H$, $\zeta = \zeta'$ and $|H(x)| \leq n$

(7) $[\![x \geq n]\!]^{\langle G[\zeta],H[\zeta']\rangle} = \mathbb{T}$ iff $G = H$, $\zeta = \zeta'$ and $|H(x)| \geq n$

(8) $[\![\phi \wedge \psi]\!]^{\langle G[\zeta],H[\zeta']\rangle} = \mathbb{T}$ iff there is a K and a ζ'' s.t. $[\![\phi]\!]^{\langle G[\zeta],K[\zeta'']\rangle} = \mathbb{T}$ and $[\![\psi]\!]^{\langle K[\zeta''],H[\zeta']\rangle} = \mathbb{T}$

(9) $[\![\phi \vee \psi]\!]^{\langle G[\zeta],H[\zeta']\rangle} = \mathbb{T}$ iff $G = H$, $\zeta = \zeta'$ and there is a K and a ζ'' s.t. $[\![\phi]\!]^{\langle G[\zeta],K[\zeta'']\rangle} = \mathbb{T}$ or $[\![\psi]\!]^{\langle G[\zeta],K[\zeta'']\rangle} = \mathbb{T}$

(10) Random assignment: $[\![[x]]\!]^{\langle G[\zeta],H[\zeta']\rangle} = \mathbb{T}$ iff $H[x]G$ and $\zeta = \zeta'$

(11) $[\![\mathbf{M}(\phi)]\!]^{\langle G[\zeta],H[\zeta']\rangle} = \mathbb{T}$ iff $[\![\phi]\!]^{\langle G[\zeta],H[\zeta']\rangle} = \mathbb{T}$ and there is no H' s.t. $H \subsetneq H'$ and $[\![\phi]\!]^{\langle G[\zeta],H'[\zeta']\rangle} = \mathbb{T}$

(12) $[\![\mathbf{M}x(\phi)]\!]^{\langle G[\zeta],H[\zeta']\rangle} = \mathbb{T}$ iff $[\![[x] \wedge \phi]\!]^{\langle G[\zeta],H[\zeta']\rangle} = \mathbb{T}$ and there is no H' s.t. $H(x) \subsetneq H'(x)$ and $[\![[x] \wedge \phi]\!]^{\langle G[\zeta],H'[\zeta']\rangle} = \mathbb{T}$

(13) ϕ is a test iff for any sets of assignments G and H and any sets of formulas ζ and ζ', if $[\![\phi]\!]^{\langle G[\zeta],H[\zeta']\rangle} = \mathbb{T}$, then $G = H$ and $\zeta = \zeta'$

(14) $[\![^{\phi}]\!]^{\langle G[\zeta],H[\zeta']\rangle} = \mathbb{T}$ iff ϕ is a test, $G = H$ and $\zeta' = \zeta \cup \{\phi\}$

(15) $[\![\delta(\phi)]\!]^{\langle G[\zeta],H[\zeta']\rangle} = \mathbb{T}$ iff $\zeta = \zeta'$ and there exists a relation $\mathcal{R}$ between assignments and sets of assignments, i.e., of the form $\mathcal{R}(g, K)$, s.t.:

a. $G = \mathbf{Dom}(\mathcal{R})$ and $H = \bigcup \mathbf{Ran}(\mathcal{R})$

b. for all g and all K such that $\mathcal{R}(g, K)$, there is a (possibly empty) set of formulas $\{\psi_1, \ldots, \psi_m\}$ s.t. $[\![\phi]\!]^{\langle \{g\}[\zeta],K[\zeta\cup\{\psi_1,\ldots,\psi_m\}]\rangle} = \mathbb{T}$ and $[\![\psi_1 \wedge \ldots \wedge \psi_m]\!]^{\langle K[\zeta],K[\zeta]\rangle} = \mathbb{T}$.

(16) Truth: a formula ϕ is true relative to an input context $G[\emptyset]$ iff there is an output set of assignments H and a (possibly empty) set of tests $\{\psi_1, \ldots, \psi_m\}$ s.t. $[\![\phi]\!]^{\langle G[\emptyset],H[\{\psi_1,\ldots,\psi_m\}]\rangle} = \mathbb{T}$ and $[\![\psi_1 \wedge \ldots \wedge \psi_m]\!]^{\langle H[\emptyset],H[\emptyset]\rangle} = \mathbb{T}$.